JOURNAL OF DIFFERENTIAL EQUATIONS **157**, 247–318 (1999)
Article ID jdeq.1998.3590, available online at http://www.idealibrary.com on IDEAL®

# Codimension-One Riemann Solutions: Classical Missing Rarefaction Cases

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Received April 22, 1997; revised May 16, 1997

This paper is the third in a series that undertakes a systematic investigation of Riemann solutions of systems of two conservation laws in one spatial dimension.



pairs), constituting the most classical degeneracies, are studied in detail. Precise conditions for a codimension-one degeneracy are identified in each case, as are conditions for folding of the Riemann solution surface, which can occur in 4 of the cases. Such folding gives rise to local multiplicity or nonexistence of Riemann solutions. 

*Key Words:* conservation law; Riemann problem; viscous profile.

## 1. INTRODUCTION

We consider systems of two conservation laws in one space dimension, partial differential equations of the form

$$U_t + F(U)_x = 0 \tag{1.1}$$

with $t > 0$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ a smooth map. The most basic initial-value problem for Eq. (1.1) is the *Riemann problem*, in which the initial data are piecewise constant with a single jump at $x = 0$:

$$U(x, 0) = \begin{cases} U_L & \text{for} \quad x < 0, \\ U_R & \text{for} \quad x > 0. \end{cases} \tag{1.2}$$

This paper is the third in a series of papers in which we study the structure of solutions of Riemann problems.

[1] This work was supported in part by the National Science Foundation under grants DMS-9205535 and DMS-9501255.



0022-0396/99 $30.00

We seek piecewise continuous weak solutions of Riemann problems in the scale-invariant form $U(x, t) = \hat{U}(x/t)$ consisting of a finite number of constant parts, continuously changing parts (*rarefaction waves*), and jump discontinuities (*shock waves*). Shock waves occur when

$$\lim_{\xi \to s-} \hat{U}(\xi) = U_- \neq U_+ = \lim_{\xi \to s+} \hat{U}(\xi). \tag{1.3}$$

They are required to satisfy the following *viscous profile admissibility criterion*: a shock wave is admissible provided that the ordinary differential equation

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \tag{1.4}$$

has a heteroclinic solution, or a finite sequence of such solutions, leading from the equilibrium $U_-$ to a second equilibrium $U_+$.

By the term *Riemann solution* for Eqs. (1.1) and (1.2) we mean a weak solution $U$ of this kind (or, equivalently, the scale-invariant function $\hat{U}$, or the sequence of waves in $U$, or the quadruple $(\hat{U}, U_L, U_R, F)$). There are various *types* of rarefaction and shock waves (e.g., 1-family rarefaction waves and classical 1-family shock waves); the *type* of a Riemann solution is the sequence of types of its waves.

Our approach to understanding Riemann solutions is to investigate the local structure of the set of Riemann solutions: we consider a particular $(\hat{U}^*, U_L^*, U_R^*, F^*)$ and construct nearby ones. More precisely, we define an open neighborhood $\mathscr{X}$ of $\hat{U}^*$ in a Banach space of scale-invariant functions $\hat{U}$, open neighborhoods $\mathscr{U}_L$ and $\mathscr{U}_R$ of $U_L^*$ and $U_R^*$ in $\mathbb{R}^2$, respectively, and an open neighborhood $\mathscr{B}$ of $F^*$ in a Banach space of smooth flux functions $F$. Then our goal is to construct a set $\mathscr{R}$ of Riemann solutions $(\hat{U}, U_L, U_R, F) \in \mathscr{X} \times \mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ near $(\hat{U}^*, U_L^*, U_R^*, F^*)$. To guide this construction, we view $\mathscr{R}$ as organized into strata of successively higher codimension.

The largest stratum of $\mathscr{R}$, which has codimension zero within $\mathscr{R}$, consists of structurally stable Riemann solutions. For such solutions, $\hat{U}$ changes continuously, and its type remains unchanged, when $(U_L, U_R, F)$ varies in certain open subsets of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. Moreover, the left and right states and speeds of each wave in $\hat{U}$ depend smoothly on $(U_L, U_R, F)$.

In Ref. [9], we identified a set of sufficient conditions for structural stability of strictly hyperbolic Riemann solutions. Briefly, these conditions have the following character.

(H0) There is a restriction on the sequence of wave types in the solution.

(H1) Each wave satisfies certain nondegeneracy conditions.

(H2) The "wave group interaction condition" is satisfied. In the simplest case, the forward wave curve and the backward wave curve are transverse.

(H3) If a shock wave represented by a connection *to* a saddle is followed by another represented by a connection *from* a saddle, the shock speeds differ.

The methods by which these conditions were derived strongly suggest that they are also necessary for structural stability.

In Ref. [10] we began an investigation of the Riemann solutions that occur when one passes to the boundary of the set of structurally stable strictly hyperbolic Riemann solutions by violating a single condition on this list, but the Riemann solution remains strictly hyperbolic. Under appropriate nondegeneracy conditions, these Riemann solutions constitute a graph over a codimension-one submanifold of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$.

A point $(\hat{U}^*, U_L^*, U_R^*, F^*)$ represents a *codimension-one Riemann solution* if there exists a codimension-one submanifold $\mathscr{S}$ of $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ with the following properties. For each point $(U_L, U_R, F) \in \mathscr{S}$ near $(U_L^*, U_R^*, F^*)$, there is a structurally unstable Riemann solution $\hat{U}$ near $\hat{U}^*$ such that (1) $\hat{U}$ has the same type as $\hat{U}^*$ and (2) the endpoints and speeds of each wave in $\hat{U}$ depend smoothly on $(U_L, U_R, F)$. Furthermore, (3) $\mathscr{S}$ bounds a region in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ that corresponds to structurally stable solutions. In particular, we obtain a set $\mathscr{R}$ of Riemann solutions as a graph of a function from a manifold-with-boundary in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$ to $\mathscr{X}$. Finally, (4) $\mathscr{S}$ is situated with a certain regularity in $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$: either $\mathscr{S}$ is in general position relative to planes of constant $(U_L, F)$ and planes of constant $(U_R, F)$, so that $(U_L, F)$ and $(U_R, F)$ both serve as good coordinates for $\mathscr{S}$; or $\mathscr{S}$ is a cylinder over a hypersurface in $(U_L, F)$-, $(U_R, F)$-, or $F$-space.

The codimension-one submanifolds of structurally unstable solutions in $\mathscr{R}$ that arise in this manner can be classified in several ways.

(A) The codimension-one submanifolds of $\mathscr{R}$ can be classified with respect to how they are situated in $\mathscr{R}$.

We distinguish:

(1) *Joins.* $\mathscr{R}$ is formed from two manifolds-with-boundary joined along their common boundary. As the boundary is crossed, a structurally stable Riemann solution becomes degenerate and then turns into a structurally stable solution of a different type.

(2) *Folds.* $\mathscr{R}$ is a manifold homeomorphic to $\mathbb{R}^4 \times \mathscr{B}$, and there is no change in type of the Riemann solution upon crossing the codimension-one submanifold, but there is a fold in the projection to $\mathscr{U}_L \times \mathscr{U}_R \times \mathscr{B}$. Thus $\mathscr{R}$ fails to be a graph over $(U_L, U_R, F)$-space.

(3) *Frontiers.* $\mathscr{R}$ is a manifold-with-boundary homeomorphic to $\mathbb{R}^3 \times \mathbb{R}_+ \times \mathscr{B}$. Riemann solutions exist only on one side of the codimension-one submanifold.

(B) The codimension-one submanifolds of $\mathscr{R}$ can be classified with respect to how $\mathscr{S}$ is situated in $\mathbb{R}^4 \times \mathscr{B}$. Let $(U_L^*, U_R^*, F^*) \in \mathscr{S}$. Four possibilities occur:

(1) *Intermediate boundary.* The submanifold $\mathscr{S}$ is transverse to both of the two-dimensional planes $\{(U_L, U_R, F): (U_L, F) = (U_L^*, F^*)\}$ and $\{(U_L, U_R, F): (U_R, F) = (U_R^*, F^*)\}$. Thus for each $(U_L, F)$ near $(U_L^*, F^*)$, $\mathscr{S}$ meets the corresponding copy of the $U_R$-plane in a curve; and for each $(U_R, F)$ near $(U_R^*, F^*)$, $\mathscr{S}$ meets the corresponding copy of the $U_L$-plane in a curve. In other words, if $U_L$ and $F$ are fixed, codimension-one Riemann solutions correspond to a curve in the $U_R$-plane; and if $U_R$ and $F$ are fixed, they correspond to a curve in the $U_L$-plane.

(2) *$U_L$-boundary.* There is a codimension-one submanifold $\tilde{\mathscr{S}}$ in $(U_L, F)$-space, transverse to the two-dimensional plane $\{(U_L, F): F = F^*\}$, such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $(U_L, F) \in \tilde{\mathscr{S}}$. Thus for each $F$ near $F^*$ there is a curve $\mathscr{C}(F)$ in the $U_L$-plane such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $U_L \in \mathscr{C}(F)$. That is, for a specific system of conservation laws, codimension-one Riemann solutions occur when $U_L$ lies on a fixed curve. Another type of boundary is obtained through duality by reversing the roles of $U_L$ and $U_R$ in this definition.

(3) *F-boundary.* There is a codimension-one submanifold $\tilde{\mathscr{S}}$ in $\mathscr{B}$ such that $(U_L, U_R, F) \in \mathscr{S}$ if and only if $F \in \tilde{\mathscr{S}}$.

(C) The codimension-one Riemann solutions can be classified with respect to the number of solutions of nearby Riemann problems. In the case of a fold, for data $(U_L, U_R, F)$ on one side of $\mathscr{S}$, there are two nearby structurally stable Riemann solutions; for data in $\mathscr{S}$, there is a locally unique codimension-one solution; and for data on the other side of $\mathscr{S}$, there is no nearby Riemann solution. The same situation can occur along some of the Riemann solution joins. In classical examples, the two manifolds-with-boundary, which meet along their common boundary, project to different sides of $\mathscr{S}$, so that there is local existence and uniqueness of Riemann solutions. It is possible, however, for the two manifolds-with-boundary to project to the same side of $\mathscr{S}$, so that for nearby data there are two, one, or zero nearby Riemann solutions, as in the case of a fold. For a frontier, there is a locally unique solution on $\mathscr{S}$ and on one side of $\mathscr{S}$, but no solution on the other side.

In Ref. [9] we noted that violations of (H0) that occur when one passes to the boundary of the set of structurally stable strictly hyperbolic Riemann solutions can be identified with violations of (H1). We then amalgamated all violations of hypothesis (H2) into a single case; we amalgamated violations of (H1) and (H3) that are analogous under a duality between slow and fast waves; we dropped from consideration violations of (H1) that lead to failure of strict hyperbolicity, or that have codimension higher than one. We found that there were 63 remaining violations of (H0)–(H3), each of which appears to give rise, under appropriate nondegeneracy conditions, to codimension-one Riemann solutions. Indeed, most occur in the literature. We noted that of these 63 degeneracies, four are folds; five are frontiers; and 54 form 27 pairs of related degeneracies that give rise to 27 joins.

We did not give detailed proofs of any of these facts.

In this paper we begin to study in detail the 15 "missing slow rarefaction" degeneracies, together with the 15 degeneracies that pair with them to produce Riemann solution joins. These degeneracies constitute 30 of the 63 degeneracies of Ref. [10]. The 15 pairs are listed in Table 5.1 of Ref. [10]. A missing rarefaction solution is a Riemann solution in which (H0) is violated because a rarefaction wave is missing; alternatively, (H1) is violated because the length of a rarefaction has shrunk to zero. (There are fifteen dual cases of missing fast rarefactions, which are completely analagous.) Nine of the 15 missing slow rarefaction cases are classical, in that the shocks adjacent to the missing rarefactions almost satisfy the Lax criterion. We shall study these cases here, together with the nine degeneracies that pair with them.

Six of the nine missing rarefaction degeneracies studied in this paper can give rise to intermediate boundaries. For each of them, we give the precise conditions under which the degeneracy gives rise to a codimension-one Riemann solution that lies in an intermediate boundary. Each of these six degeneracies can give rise to other types of boundaries when followed by a 1-rarefaction somewhere later in the wave sequence, but we do not study these possibilities here. The other three missing rarefaction degeneracies studied in this paper cannot give rise to intermediate boundaries, but can give rise to $U_L$-boundaries. For each of them, we give the precise conditions under which the degeneracy gives rise to a codimension-one Riemann solution that lies in a $U_L$-boundary. One of these degeneracies can give rise to an $F$-boundary when it is preceded somewhere in the wave sequence by a 2-rarefaction, but we do not study this possibility here.

The question of folding at Riemann solution joins was not addressed in Ref. [10]. In this paper we show that for four of the nine missing rarefaction cases under study, under atypical but open conditions, such folding occurs.

The degeneracies studied in this paper can all arise in the slow wave group of the Riemann solution; most can also arise in later wave groups. (Their duals can all arise in the fast wave group of the Riemann solution.)

Complicated Riemann solutions were constructed in the context of scalar conservation laws by Oleĭnik [7]. These complications are analagous to those that can occur in the slow or fast wave group of a Riemann solution for two conservation laws. In [12], Wendroff showed the smoothness of slow and fast composite wave curves at certain junction points. In [1], Dafermos related these complicated Riemann solutions to similarity solutions of conservation laws with a certain time-dependent viscosity term. Furtado [3] gave a general discussion of the construction of slow and fast wave curves in the context of the Lax admissibility criterion, which unfortunately allows splitting of wave curves. Composite slow and fast waves were used by Liu [6] to study Riemann solutions near curves where genuine nonlinearity fails, and by Isaacson *et al.* [4], Shearer *et al.* [11], and Schaeffer and Shearer [8] to study Riemann solutions near an umbilic point. Reference [11] includes a bifurcation theory approach to shock waves, which had been initiated by Foy [2]; Ref. [8] discusses $U_L$- and intermediate boundaries (there called $U_R$-boundaries). Despite the extensive literature on the complications that can arise in slow and fast wave groups (of which we have cited only a representative part), there does not seem to be a systematic study of the type presented in this paper.

The remainder of the paper is organized as follows. In Sections 2 and 3 we review terminology and results about structurally stable Riemann solutions and codimension-one Riemann solutions from Refs. [9, 10]. In Section 4 we explain the general approach we will take to analyzing the various degeneracies, and we summarize the results. In Sections 5–13 we treat in detail the nine classical missing rarefaction cases. These sections can be read independently. Each of Sections 6–13 includes vector field bifurcation diagrams that contain the essence of the degeneracy. Some final remarks are in Section 14.

## 2. BACKGROUND ON STRUCTURALLY STABLE RIEMANN PROBLEM SOLUTIONS

We consider the system (1.1) with $t \in \mathbb{R}_+$, $x \in \mathbb{R}$, $U(x, t) \in \mathbb{R}^2$, and $F: \mathbb{R}^2 \to \mathbb{R}^2$ a $C^2$ map. Let

$$\mathscr{U}_F = \{ U \in \mathbb{R}^2 : DF(U) \text{ has distinct real eigenvalues} \} \tag{2.1}$$

be the *strictly hyperbolic region* in state-space. We shall call a Riemann solution $\hat{U}$ *strictly hyperbolic* if $\hat{U}(\xi) \in \mathscr{U}_{\mathscr{F}}$ for all $\xi \in \mathbb{R}$. In this paper, all Riemann solutions are assumed to be strictly hyperbolic. For $U \in \mathscr{U}_F$, let $\lambda_1(U) < \lambda_2(U)$ denote the eigenvalues of $DF(U)$, and let $\ell_i(U)$ and

$r_i(U)$, $i=1,2$, denote corresponding left and right eigenvectors with $\ell_i(U)\, r_j(U) = \delta_{ij}$.

A *rarefaction wave* of *type* $R_i$ is a differentiable map $\hat{U}\colon [a, b] \to \mathscr{U}_F$, where $a < b$, such that $\hat{U}'(\xi)$ is a multiple of $r_i(\hat{U}(\xi))$ and $\xi = \lambda_i(\hat{U}(\xi))$ for each $\xi \in [a, b]$. The states $U = \hat{U}(\xi)$ with $\xi \in [a, b]$ comprise the *rarefaction curve* $\bar{\Gamma}$. The definition of rarefaction wave implies that if $U \in \bar{\Gamma}$, then

$$D\lambda_i(U)\, r_i(U) = \ell_i(U)\, D^2F(U)(r_i(U), r_i(U)) \neq 0. \tag{2.2}$$

Condition (2.2) is *genuine nonlinearity* of the *i*th characteristic line field at $U$. Assuming (2.2), we can choose $r_i(U)$ such that

$$D\lambda_i(U)\, r_i(U) = 1. \tag{2.3}$$

In this paper we shall assume this has been done wherever (2.2) is satisfied. The definition also implies that $\lambda_i(U_-) < \lambda_i(U_+)$, where $U_- = U(a)$ and $U_+ = U(b)$ are the *left* and *right states* of the rarefaction wave, respectively. We will find it convenient to associate a specific *speed* $s$ to a rarefaction wave: for a rarefaction wave of type $R_1$, $s = \lambda_1(U_+)$; for a rarefaction wave of type $R_2$, $s = \lambda_2(U_-)$.

A *shock wave* consists of a *left state* $U_- \in \mathscr{U}_F$, a *right state* $U_+ \in \mathscr{U}_F$, a *speed* $s$, and a *connecting orbit* $\Gamma$, i.e., an orbit of the ordinary differential equation (1.4). For any equilibrium $U \in \mathscr{U}_F$ of Eq. 1.4, the eigenvalues of the linearization at $U$ are $\lambda_i(U) - s$, $i = 1, 2$. We shall use the terminology defined in Table I for such an equilibrium. The *type* of a shock wave is determined by the equilibrium types of its left and right states. (For example, $w$ is of type $R \cdot S$ if its connecting orbit joins a repeller to a saddle.)

An *elementary wave* $w$ is either a rarefaction wave or a shock wave. We write

$$w : U_- \xrightarrow{s} U_+$$

if $w$ has left state $U_-$, right state $U_+$, and speed $s$. Note that an elementary wave also has a *type* $T$, as defined above.

TABLE I

Types of Equilibria

| Name | Symbol | Eigenvalues | |
|---|---|---|---|
| Repeller | $R$ | + | + |
| Repeller-saddle | $RS$ | 0 | + |
| Saddle | $S$ | − | + |
| Saddle-attractor | $SA$ | − | 0 |
| Attractor | $A$ | − | − |

Associated with each elementary wave is a *speed interval* $\sigma$: for a rarefaction wave of type $R_i$, $\sigma = [\lambda_i(U_-), \lambda_i(U_+)]$, whereas for a shock wave of speed $s$, $\sigma = [s, s]$. If $\sigma_1$ and $\sigma_2$ are speed intervals, we write $\sigma_1 \leqslant \sigma_2$ if $s_1 \leqslant s_2$ for every $s_1 \in \sigma_1$ and $s_2 \in \sigma_2$. Also associated with each elementary wave is the set $\bar{\Gamma}$: if $w$ is a rarefaction wave, $\bar{\Gamma}$ denotes its rarefaction curve; if $w$ is a shock wave, then $\bar{\Gamma}$ denotes the closure of its connecting orbit. We shall say that an open set $\mathcal{N} \subseteq \mathbb{R}^2$ is a *neighborhood* of the elementary wave $w$: $U_- \xrightarrow{s} U_+$ if $\bar{\Gamma} \subset \mathcal{N}$.

Sequences of elementary waves can be used to construct solutions of Riemann problems. A wave sequence $(w_1, w_2, ..., w_n)$ is said to be *allowed* if:

(W1) for each $i = 1, ..., n-1$, the right state of $w_i$ coincides with the left state of $w_{i+1}$;

(W2) the speed intervals $\sigma_i$ for $w_i$ satisfy

$$\sigma_1 \leqslant \sigma_2 \leqslant \cdots \leqslant \sigma_n; \tag{2.4}$$

(W3) no two successive waves are rarefaction waves of the same type.

For such a wave sequence we write

$$(w_1, w_2, ..., w_n): U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n. \tag{2.5}$$

The *type* of $(w_1, w_2, ..., w_n)$ is $(T_1, T_2, ..., T_n)$ if $w_i$ has type $T_i$. If $U_0 = U_L$ and $U_n = U_R$, then associated with an allowed wave sequence $(w_1, w_2, ..., w_n)$ is a solution $U(x, t) = \hat{U}(x/t)$ of the Riemann problem (1.1)–(1.2). Therefore we shall often refer to an allowed wave sequence as a *Riemann solution.*

Let

$$(w_1^*, w_2^*, ..., w_n^*): U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{2.6}$$

be a Riemann solution for $U_t + F^*(U)_x = 0$. Fix a compact set $K \subset \mathbb{R}^2$ such that Int $K$ is a neighborhood of $w_i^*$ for $i = 1, ..., n$. Let $\mathcal{B}$ denote the Banach space of $C^2$ functions $F: K \to \mathbb{R}^2$, equipped with the $C^2$ norm. Also, let $\mathcal{H}(\text{Int } K)$ denote the set of nonempty, closed subsets of Int $K$, which we equip with the Hausdorff metric.

Definition 2.1. We shall say that the Riemann solution (2.6) is *structurally stable* if there are neighborhoods $\mathcal{U}_i$ of $U_i^*$, $\mathcal{I}_i$ of $s_i^*$, and $\mathcal{F}$ of $F^*$ and a $C^1$ map

$$G: \mathcal{U}_0 \times \mathcal{I}_1 \times \mathcal{U}_1 \times \mathcal{I}_2 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{3n-2}$$

with $G(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)=0$ such that:

(P1) $G(U_0, s_1, U_1, s_2, ..., s_n, U_n, F)=0$ implies that there exists a Riemann solution

$$(w_1, w_2, ..., w_n)\colon U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$$

for $U_t+F(U)_x=0$ with successive waves of the same types as those of the wave sequence (2.6) and with each $w_i$ contained in Int $K$;

(P2) $DG(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0=0=\dot{U}_n, \dot{F}=0\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Condition (P2) implies, by the implicit function theorem, that $G^{-1}(0)$ is a graph over $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$; $(s_1, U_1, ..., U_{n-1}, s_n)$ is determined by $(U_0, U_n, F)$. Therefore for each wave $w_i$ we can define a map $\bar{\Gamma}_i\colon \mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F} \to \mathscr{H}(\text{Int}\, K)$; namely, $\bar{\Gamma}_i(U_0, U_n, F)$ is the rarefaction curve or the closure of the connecting orbit of the wave $w_i$. We further require that

(P3) $(w_1, w_2, ..., w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

The map $G$ will be said to *exhibit* the structural stability of the Riemann solution (2.6).

Associated with each type of elementary wave is a *local defining map*, which we use to construct maps $G$ that exhibit structural stability. Let $w^*\colon U_-^* \xrightarrow{s^*} U_+^*$ be an elementary wave of type $T$ for $U_t+F^*(U)_x=0$. The local defining map $G_T$ has as its domain a set of the form $\mathscr{U}_- \times \mathscr{I} \times \mathscr{U}_+ \times \mathscr{F}$ (with $\mathscr{U}_\pm$ being neighborhoods of $U_\pm^*$, $\mathscr{I}$ a neighborhood of $s^*$, and $\mathscr{F}$ a neighborhood of $F^*$). The range is some $\mathbb{R}^e$; the number $e$ depends only on the wave type $T$. The local defining map is such that $G_T(U_-^*, s^*, U_+^*, F^*)=0$. Moreover, if certain *wave non-degeneracy conditions* are satisfied at $(U_-^*, s^*, U_+^*, F^*)$, then there is a neighborhood $\mathscr{N}$ of $w^*$ such that:

(D1) $G_T(U_-, s, U_+, F)=0$ if and only if there exists an elementary wave $w\colon U_- \xrightarrow{s} U_+$ of type $T$ for $U_t+F(U)_x=0$ contained in $\mathscr{N}$;

(D2) $DG_T(U_-^*, s^*, U_+^*, F^*)$, restricted to the space $\{(\dot{U}_-, \dot{s}, \dot{U}_+, \dot{F})\colon \dot{F}=0\}$, is surjective.

Condition (D2) implies, by the implicit function theorem, that $G_T^{-1}(0)$ is a manifold of codimension $e$. Therefore we can define a map $\bar{\Gamma}$ from this manifold to $\mathscr{H}(\operatorname{Int} K)$ (just as above). In fact,

(D3) $w$ can be chosen so that $\bar{\Gamma}$ is continuous.

We now discuss local defining maps and nondegeneracy conditions for the types of elementary waves that occur in this paper.

Let

$$\mathscr{U}_1 = \{ U \in \mathscr{U} : D\lambda_1(U)\, r_1(U) \neq 0 \}.$$

In $\mathscr{U}_1$ we can assume that Eq. (2.3) holds with $i = 1$. For each $U_- \in \mathscr{U}_1$, define $\psi$ to be the solution of

$$\frac{\partial \psi}{\partial s}(U_-, s) = r_1(\psi(U_-, s)),$$

$$\psi(U_-, \lambda_1(U_-)) = U_-.$$

By (2.3), if $\psi(U_-, s) = U$, then $s = \lambda_1(U)$. Thus there is a rarefaction wave of type $R_1$ for $U_t + F(U)_x = 0$ from $U_-$ to $U_+$ with speed $s$ if and only if

$$U_+ - \psi(U_-, s) = 0 \tag{2.7}$$

$$s = \lambda_1(U_+) > \lambda_1(U_-). \tag{2.8}$$

Equations (2.7) are defining equations for rarefaction waves of types $R_1$. The nondegeneracy conditions for rarefaction waves of type $R_1$, which are implicit in our definition of rarefaction, are the speed inequality (2.8), and the genuine nonlinearity condition (2.2).

For future reference we state:

Lemma 2.2.

$$D\psi(U_-, \lambda_1(U_-))(ar_1(U_-) + br_2(U_-), \dot{s})$$
$$= (\dot{s} - bD\lambda_1(U_-)\, r_2(U_-))\, r_1(U_-) + br_2(U_-).$$

*Proof.* Let $\chi(U_-, t)$ be the flow of $\dot{U} = r_1(U)$, so that

$$\chi(U_-, 0) = U_-$$

$$\frac{\partial \chi}{\partial t}(U_-, t) = r_1(\chi(U_-, t)).$$

Then

$$\psi(U_-, s) = \chi(U_-, s - \lambda_1(U_-)).$$

Therefore

$$\begin{aligned} D_1\psi(U_-, \lambda_1(U_-))\,\dot{U}_- &= D_1\chi(U_-, 0)\,\dot{U}_- - \frac{\partial\chi}{\partial s}(U_-, 0)\, D\lambda_1(U_-)\,\dot{U}_- \\ &= (I - r_1(U_-)\, D\lambda_1(U_-))\,\dot{U}_-, \end{aligned} \tag{2.9}$$

$$\frac{\partial\psi}{\partial s}(U_-, \lambda_1(U_-)) = \frac{\partial\chi}{\partial s}(U_-, 0) = r_1(U_-).$$

Using Eq. (2.3), we have

$$\begin{aligned} &D\psi(U_-, \lambda_1(U_-))(ar_1(U_-) + br_2(U_-), \dot{s}) \\ &\quad = (I - r_1(U_-)\, D\lambda_1(U_-))(ar_1(U_-) + br_2(U_-)) + \dot{s}r_1(U_-) \\ &\quad = (\dot{s} - b\, D\lambda_1(U_-)\, r_2(U_-))\, r_1(U_-) + br_2(U_-). \quad \blacksquare \end{aligned}$$

Next we consider shock waves. If there is to be a shock wave solution of $U_t + F(U)_x = 0$ from $U_-$ to $U_+$ with speed $s$, we must have that

$$F(U_+) - F(U_-) - s(U_+ - U_-) = 0; \tag{E0}$$

$$\dot{U} = F(U) - F(U_-) - s(U - U_-) \text{ has an orbit from } U_- \text{ to } U_+. \tag{C0}$$

The two-component equation (E0) is a defining equation. In the context of structurally stable Riemann solutions, condition (C0) is an open condition, and therefore is regarded as a nondegeneracy condition, for all but transitional shock waves, which do not occur in this paper.

In Table II we list additional defining equations and nondegeneracy conditions for the types of shock waves that occur in this paper. The wave nondegeneracy conditions are open conditions. Conditions (C1)–(C2) are that the connection $\Gamma$ is not *distinguished*; for $RS \cdot S$ and $RS \cdot RS$ shock waves, this means that the connection $\Gamma$ should not lie in the unstable manifold of $U_-$ (i.e., the unique invariant curve tangent to an eigenvector with positive eigenvalue).

For the Riemann solution (2.6), let $w_i^*$ have type $T_i$ and local defining map $G_{T_i}$, with range $\mathbb{R}^{e_i}$. For appropriate neighborhoods $\mathcal{U}_i$ of $U_i^*$, $\mathcal{I}_i$ of $s_i^*$, $\mathcal{F}$ of $F^*$, and $\mathcal{N}_i$ of $w_i^*$, we can define a map $G\colon \mathcal{U}_0 \times \mathcal{I}_1 \times \cdots \times \mathcal{I}_n \times \mathcal{U}_n \times \mathcal{F} \to \mathbb{R}^{e_1 + \cdots + e_n}$ by $G = (G_1, ..., G_n)$, where

$$G_i(U_0, s_1, ..., s_n, U_n, F) = G_{T_i}(U_{i-1}, s_i, U_i, F).$$

TABLE II

Additional Defining Equations and Nondegeneracy Conditions for Slow Shock Waves

| Type of shock | Additional defining equations | | Nondegeneracy conditions | |
|---|---|---|---|---|
| $R \cdot S$ | None | | None | |
| $R \cdot RS$ | $\lambda_1(U_+) - s = 0$ | (E1) | $D\lambda_1(U_+)\, r_1(U_+) \neq 0$ | (G1) |
| | | | $\ell_1(U_+)(U_+ - U_-) \neq 0$ | (B1) |
| $RS \cdot S$ | $\lambda_1(U_-) - s = 0$ | (E2) | $D\lambda_1(U_-)\, r_1(U_-) \neq 0$ | (G2) |
| | | | not distinguished connection | (C1) |
| $RS \cdot RS$ | $\lambda_1(U_-) - s = 0$ | (E3) | $D\lambda_1(U_-)\, r_1(U_-) \neq 0$ | (G3) |
| | $\lambda_1(U_+) - s = 0$ | (E4) | $D\lambda_1(U_+)\, r_1(U_+) \neq 0$ | (G4) |
| | | | $\ell_1(U_+)(U_+ - U_-) \neq 0$ | (B2) |
| | | | not distinguished connection | (C2) |

The map $G$ is called the *local defining map* of the wave sequence (2.6). Assuming the wave nondegeneracy conditions, if $G(U_0, s_1, ..., s_n, U_n, F) = 0$, then for each $i = 1, ..., n$, there is an elementary wave $w_i$: $U_{i-1} \xrightarrow{s_i} U_i$ of type $T_i$ for $U_t + F(U)_x = 0$ contained in $\mathcal{N}_i$, for which $\bar{\Gamma}_i$ is continuous.

In view of the requirement that the local defining map have range $\mathbb{R}^{3n-2}$, a necessary condition for $G = (G_1, ..., G_n)$ to exhibit the structural stability of the wave sequence (2.6) is that

$$\sum_{i=1}^{n} e_{\mathrm{i}} = 3n - 2, \tag{2.10}$$

i.e.,

$$\sum_{i=1}^{n} (3 - e_i) = 2. \tag{2.11}$$

We are therefore led to define the *Riemann number* of an elementary wave type $T$ to be

$$\rho(T) = 3 - e(T),$$

where $e(T)$ is the number of defining equations for a wave of type $T$. For convenience, if $w$ is an elementary wave of type $T$, we shall write $\rho(w)$ instead of $\rho(T)$. Because of (2.11) we concentrate our attention on allowed sequences of elementary waves $(w_1, ..., w_n)$ with $\sum_{i=1}^{n} \rho(w_i) = 2$.

To describe the results of Ref. [9], we need the following definitions.

A 1-*wave group* is either a single $R \cdot S$ wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1 RS \cdot RS) \cdots (R_1 RS \cdot RS)\, R_1(RS \cdot S), \tag{2.12}$$

where the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite.*

A *transitional wave group* is either a single $S \cdot S$ wave or an allowed sequence of elementary waves of the form

$$(R \cdot RS)(R_1 RS \cdot RS) \cdots (R_1 RS \cdot RS)\, R_1\, (RS \cdot S), \tag{2.13}$$

or

$$(S \cdot SA)\, R_2(SA \cdot SAR_2) \cdots (SA \cdot SAR_2)\, SA \cdot S, \tag{2.14}$$

the terms in parentheses being optional. In cases (2.13) and (2.14), the group is termed *composite.*

A 2-*wave group* is either a single $S \cdot A$ wave or an allowed sequence of elementary waves of the form

$$(S \cdot SA)\, R_2(SA \cdot SAR_2) \cdots (SA \cdot SAR_2)(SA \cdot A), \tag{2.15}$$

where again the terms in parentheses are optional. If any of the terms in parentheses are present, the group is termed *composite.*

In Ref. [9] the following are proved.

THEOREM 2.3 (Wave Structure). *Let* (2.6) *be an allowed sequence of elementary waves. Then*

1. $\sum_{i=1}^{n} \rho(w_i^*) \leqslant 2$;
2. $\sum_{i=1}^{n} \rho(w_i^*) = 2$ *if and only if the following conditions are satisfied.*

(1) *Suppose that the wave sequence* (2.6) *includes no* $SA \cdot RS$ *waves. Then it consists of one* 1-*wave group, followed by an arbitrary number of transitional wave groups* (*in any order*), *followed by one* 2-*wave group.*

(2) *Suppose that the wave sequence* (2.6) *includes* $m \geqslant 1$ *waves of type* $SA \cdot RS$. *Then these waves separate* $m+1$ *wave sequences* $g_0, ..., g_m$. *Each* $g_i$ *is exactly as in* (1) *with the restrictions that*:

(a) *if* $i < m$, *the last wave in the group has type* $R_2$;

(b) *if* $i > 0$, *the first wave in the group has type* $R_1$.

THEOREM 2.4 (Structural Stability). *Suppose that the allowed sequence of elementary waves* (2.6) *has* $\sum_{i=1}^{n} \rho(w_i^*) = 2$. *Assume that*:

(H1) *each wave satisfies the appropriate wave nondegeneracy conditions*;

(H2) *the wave group interaction conditions, as stated precisely in Ref.* [9], *are satisfied*;

(H3) *if* $w_i^*$ *is a* $* \cdot S$ *wave and* $w_{i+1}^*$ *is an* $S \cdot *$ *wave, then* $s_i^* < s_{i+1}^*$.

*Then the wave sequence* (2.6) *is structurally stable.*

In fact, more can be concluded: not only can the connecting orbit $\Gamma_i$ of the perturbed shock wave $w_i$ be chosen to vary continuously, but also there is a neighborhood $\mathcal{N}_i$ of $w_i^*$ in which $\Gamma_i$ is unique.

Let us briefly elucidate (H2). In the absence of $SA \cdot RS$ waves, we impose one *wave group interaction condition* on how the different wave groups are related. If there are $m \geqslant 1$ waves of type $SA \cdot RS$, we impose $m+1$ wave group interaction conditions, one on each of the $m+1$ wave sequences $g_0, ..., g_m$. Roughly speaking, these conditions say that certain wave curves are transverse.

In the remainder of the paper, by a structurally stable Riemann solution we shall mean a sequence of elementary waves that satisfies the hypotheses of Theorem 2.2.

## 3. CODIMENSION-ONE RIEMANN SOLUTIONS

In order to consider conveniently codimension-one Riemann solutions, the definitions of rarefaction and shock waves in Section 2 must be generalized somewhat.

A *generalized rarefaction wave* of type $R_i$ is a continuous map $\hat{U}: [a, b] \to \mathcal{U}_{\mathcal{F}}$, where $a \leqslant b$, such that (i) the rarefaction curve $\bar{\Gamma} = \{\hat{U}(\xi): \xi \in [a, b]\}$ is an integral curve for the line field asociated to family $i$ and (ii) $\xi = \lambda_i(\hat{U}(\xi))$ for all $\xi \in [a, b]$.

A *generalized shock wave* consists of a *left state* $U_-$, a *right state* $U_+$ (possibly equal to $U_-$), a *speed* $s$, and a sequence of connecting orbits $\tilde{\Gamma}_1, \tilde{\Gamma}_2, ..., \tilde{\Gamma}_k$ of Eq. (1.4) from $U_- = \tilde{U}_0$ to $\tilde{U}_1$, $\tilde{U}_1$ to $\tilde{U}_2, ..., \tilde{U}_{k-1}$ to $\tilde{U}_k = U_+$. Note that $\tilde{U}_0$, $\tilde{U}_1, ..., \tilde{U}_k$ must be equilibria of Eq. (1.4). We allow for the possibility that $\tilde{U}_{j-1} = \tilde{U}_j$, in which case we assume that $\tilde{\Gamma}_j$ is the trivial orbit $\{\tilde{U}_j\}$.

Associated with each generalized rarefaction or generalized shock is a speed $s$, defined as before, and a curve $\bar{\Gamma}$: the rarefaction curve or the closure of $\tilde{\Gamma}_1 \cup \cdots \cup \tilde{\Gamma}_k$.

A *generalized allowed wave sequence* is a sequence of generalized rarefaction and shock waves that satisfies conditions (W1)–(W3). If $U_0 = U_L$ and

$U_n = U_R$, then associated with a generalized allowed wave sequence $(w_1, w_2, ..., w_n)$ is a solution $U(x, t) = \hat{U}(x/t)$ of the Riemann problem (1.1)–(1.2). Therefore we shall often refer to a generalized allowed wave sequence as a *Riemann solution*.

A generalized allowed wave sequence (2.6) is a *codimension-one Riemann solution* provided that there is a sequence of wave types $(T_1^*, ..., T_n^*)$ with $\sum_{i=1}^n \rho(T_i^*) = 2$, neighborhoods $\mathscr{U}_i \subseteq \mathscr{U}$ of $U_i^*$, $\mathscr{I}_i \subseteq I$ of $s_i^*$, and $\mathscr{F} \subseteq \mathscr{B}$ of $F^*$, and a $C^1$ map

$$(G, H): \mathscr{U}_0 \times \mathscr{I}_1 \times \cdots \times \mathscr{I}_n \times \mathscr{U}_n \times \mathscr{F} \to \mathbb{R}^{3n-2} \times \mathbb{R}, \tag{3.1}$$

with $G(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) = 0$ and $H(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) = 0$ such that the following conditions, (Q1)–(Q7), are satisfied.

(Q1) If $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) \geqslant 0$ then there is a generalized allowed wave sequence

$$(w_1, w_2, ..., w_n): U_0 \xrightarrow{s_1} U_1 \xrightarrow{s_2} \cdots \xrightarrow{s_n} U_n$$

for $U_t + F(U)_x = 0$ with each $w_i$ contained in Int $K$.

(Q2) If $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) > 0$, then $(w_1, w_2, ..., w_n)$ is a structurally stable Riemann solution of type $(T_1^*, ..., T_n^*)$ and $G$ exhibits its structural stability.

(Q3) If $G(U_0, s_1, ..., s_n, U_n, F) = 0$ and $H(U_0, s_1, ..., s_n, U_n, F) = 0$ then $(w_1, w_2, ..., w_n)$ is not a structurally stable Riemann solution.

(Q4) $D(G, H)(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*)$, restricted to some $(3n-1)$-dimensional space of vectors that contains $\{(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 = \dot{U}_n, \dot{F} = 0\}$, is an isomorphism.

Condition (Q4) implies, by the implicit function theorem, that $(G, H)^{-1}(0)$ is a graph over a codimension-one manifold $\mathscr{S}$ in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$, and $\mathscr{M} := (G, H)^{-1}(\{0\} \times \mathbb{R}_+)$ is a manifold-with-boundary of codimension $3n-2$. We can define maps $\bar{\Gamma}_i: \mathscr{M} \to \mathscr{H}(\text{Int } K)$. We require that

(Q5) $(w_1, w_2, ..., w_n)$ can be chosen so that each map $\bar{\Gamma}_i$ is continuous.

$(G, H)$ is again called a *local defining map*.

The surface $\mathscr{S}$ is required to be regularly situated with respect to the foliation of $U_0 U_n F$-space into planes of constant $(U_0, F)$ and planes of constant $(U_n, F)$. More precisely, let

$$\Sigma_0 = \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_n = 0 \text{ and } \dot{F} = 0\},$$

$$\Sigma_n = \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n, \dot{F}) : \dot{U}_0 = 0 \text{ and } \dot{F} = 0\}.$$

Then we require that one of the following four conditions hold:

($Q6_1$) $D(G, H)(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*)$ restricted to $\Sigma_0$ and to $\Sigma_n$, respectively, are surjective,

($Q6_2$) $D(G, H)(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*)$ restricted to $\Sigma_n$ is surjective, and there is a codimension-one manifold $\tilde{\mathscr{S}}$ through $(U_0^*, F^*)$ in $(U_0, F)$-space such that $\mathscr{S} = \mathscr{U}_n \times \tilde{\mathscr{S}}$;

($Q6_3$) $D(G, H)(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*)$ restricted to $\Sigma_0$ is surjective, and there is a codimension-one manifold $\tilde{\mathscr{S}}$ through $(U_n^*, F^*)$ in $(U_n, F)$-space such that $\mathscr{S} = \mathscr{U}_0 \times \tilde{\mathscr{S}}$;

($Q6_4$) there is a codimension-one manifold $\tilde{\mathscr{S}}$ through $F^*$ in $F$-space such that $\mathscr{S} = \mathscr{U}_0 \times \mathscr{U}_n \times \tilde{\mathscr{S}}$.

When ($Q6_1$), ($Q6_2$) or ($Q6_3$), or ($Q6_4$) holds, then the codimension-one Riemann solution is termed an *intermediate boundary*, a *$U_L$-boundary* or dual, or an *F-boundary*, respectively.

Finally, we require one of the following conditions to hold:

($Q7_1$) The linear map

$$DG(U_0^*, s_1^*, ..., s_n^*, U_n^*, F^*) \text{ restricted to } \Sigma_0 \cap \Sigma_n \tag{3.2}$$

is an isomorphism. (In this case, $\mathscr{M}$ is a smooth graph over a manifold-with-boundary in $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ with boundary $\mathscr{S}$.)

($Q7_2$) the linear map (3.4) is not surjective, but the projection of $G^{-1}(0)$ to $\mathscr{U}_0 \times \mathscr{U}_n \times \mathscr{F}$ has a fold along $(G, H)^{-1}(0, 0)$.

This case does not arise in the present paper.

A *rarefaction of zero strength* is one whose domain has zero length. A *shock of zero strength* is one with $U_L = U_R$ (and hence $\Gamma = \{U_L\}$.

A generalized allowed wave sequence is *minimal* if

(1) there are no rarefactions or shocks of zero strength;

(2) no two successive shocks have the same speed.

Among the minimal generalized allowed wave sequences we include sequences of no waves; these are given by a single $U_0 \in \mathbb{R}^2$, and represent constant solutions of Eq. (1.1).

We *shorten* a generalized allowed wave sequence by dropping a rarefaction or shock of zero strength, or by amalgamating adjacent shocks of positive strength with the same speed. Every generalized allowed wave sequence can be shortened to a unique minimal generalized allowed wave sequence. Two generalized allowed wave sequences are *equivalent* if their minimal shortenings are the same. Equivalent generalized allowed wave sequences represent the same solution $U(x, t) = \hat{U}(x/t)$ of Eq. (1.1).

Let $(U_0^*, s_1^*, U_1^*, s_2^*, ..., s_n^*, U_n^*, F^*)$ be a generalized allowed wave sequence that is a codimension-one Riemann solution of type $(T_1^*, ..., T_n^*)$. Let $\mathscr{M}$ denote the associated manifold-with-boundary, $\mathscr{M}$ being a graph over the manifold $\mathscr{S}$. Suppose there is an equivalent generalized allowed wave sequence $(U_0^\sharp, s_1^\sharp, U_1^\sharp, s_2^\sharp, ..., s_m^\sharp, U_m^\sharp, F^*)$ that is a codimension-one Riemann solution in $\partial\mathscr{N}$, $\mathscr{N} = (G^\sharp, H^\sharp)^{-1}(\{0\} \times \mathbb{R}_+)$, where Int $\mathscr{N}$ consists of structurally stable Riemann solutions of some type $(T_1^\sharp, ..., T_m^\sharp)$. Suppose in addition that $\partial\mathscr{N}$ is also a graph over $\mathscr{S}$, and the points in $\partial\mathscr{M}$ and $\partial\mathscr{N}$ above the same point in $\mathscr{S}$ are equivalent. Then the codimension-one Riemann solution (2.6) (or its equivalent generalized wave sequence) is said to lie in a *join*.

$\mathscr{M}$ and $\mathscr{N}$ are each graphs over the union of one side of $\mathscr{S}$ and $\mathscr{S}$ itself. If $\mathscr{M}$ and $\mathscr{N}$ are graphs over different sides of $\mathscr{S}$, we have a *regular join*; if $\mathscr{M}$ and $\mathscr{N}$ are graphs over the same side of $\mathscr{S}$, we have a *folded join*.

Riemann solution *frontiers* and *folds* are defined in Ref. [10], but do not occur in this paper.

Let us consider a wave sequence that satisfies (H0)–(H4) except for a degeneracy in some wave. As explained in Ref. [10], we expect:

(1) If the degeneracy follows all waves of type $R_1$ and precedes all waves of type $R_2$, it is an intermediate boundary.

(2) If the degeneracy precedes a wave of type $R_1$ and precedes all waves of type $R_2$, it is a $U_L$-boundary; the case where the degeneracy follows all waves of type $R_1$ and follows at least one wave of type $R_2$ is dual.

(3) If the degeneracy precedes a wave of type $R_1$ and follows a wave of type $R_2$, it is an $F$-boundary.

## 4. MISSING RAREFACTION SOLUTIONS: GENERAL APPROACH AND RESULTS

We wish to study Riemann solution (2.6) such that

(MR) Solution (2.6) satisfies all hypotheses of the Structural Stability Theorem, except that some rarefaction of type $R_1$ has zero strength.

If the rarefaction of zero strength is $w_j\colon U_{j-1} \xrightarrow{s_j} U_j$, then of course $U_j = U_{j-1}$, and the speed is $s_j = \lambda_1(U_{j-1})$. The rarefaction of zero strength may have a predecessor of type $R \cdot RS$, $RS \cdot RS$, $S \cdot RS$, or $SA \cdot RS$, or it may fail to have a predecessor with the same speed. Its successor may be of type $RS \cdot S$ or $RS \cdot RS$, or it may be a wave with greater speed. There are thus fifteen cases. In this paper we shall study the nine "classical" cases: there is no predecessor with the same speed, or one of type $R \cdot RS$ or $RS \cdot RS$, and the successor is one of the three given types.

Under additional nondegeneracy conditions, we shall show that such a Riemann solution (2.6) lies in a join. Our arguments will have three steps:

*Step* 1. We verify that (2.6) is a codimension-one Riemann solution.

*Step* 2. We construct a Riemann solution equivalent to (2.6) and verify that it too is a codimension-one Riemann solution.

*Step* 3. We show that the two types of codimension-one Riemann solutions are defined on the same codimension-one surface $\mathscr{S}$ in $U_0 U_n F$-space; the two types of codimension-one Riemann solutions above a given point in $\mathscr{S}$ are equivalent; and the Riemann solution join that we therefore have is of a certain type (intermediate boundary or $U_L$-boundary, regular or folded join).

All three steps will make use of the local defining map $(G, H)$ (see (3.1)). However, in the remainder of the paper, we will not show the dependence of $(G, H)$ on $F$, and we will denote the fixed flux function under consideration by $F$ rather than $F^*$.

We now discuss the three steps of our arguments in order. We begin with step 1.

We consider a wave sequence (2.6) of type $(T_1, ..., T_n)$ such that (MR) holds. In each case the local defining map $(G, H)$ is as follows:

(1) G is the map that would be used for structurally stable Riemann solutions of type $(T_1, ..., T_n)$.

(2) If the missing rarefaction is $w_j$, then

$$H(U_0, s_1, ..., s_n, U_n) = s_j - \lambda_1(U_{j-1}). \tag{4.1}$$

In order to show that (2.6) is a codimension-one Riemann solution, we must verify (Q1)–(Q7). We will first show $(\mathrm{Q7}_1)$. Since we are ignoring the dependence of $(G, H)$ on the flux function $F$, we rewrite $(\mathrm{Q7}_1)$ as

(A) $DG(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, restricted to the $(3n-2)$-dimensional space of vectors $\{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n) : \dot{U}_0 = 0 = \dot{U}_n\}$, is an isomorphism onto $\mathbb{R}^{3n-2}$.

Thus, as in the structurally stable case, the equation $G = 0$ may be solved for $(s_1, U_1, ..., U_{n-1}, s_n)$ in terms of $(U_0, U_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$. Let

$$\tilde{H}(U_0, U_n) = H(U_0, s_1(U_0, U_n), ..., s_n(U_0, U_n), U_n). \tag{4.2}$$

Properties (Q1)–(Q3) and (Q5) follow from a geometric understanding of the situation. We will discuss the geometry in each case in enough detail to make them evident. In addition, we will show that one of the following occurs:

(E1) Both $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ and $D_{U_n}\tilde{H}(U_0^*, U_n^*)$ are nonzero. Thus (Q4) and ($Q6_1$) are satisfied, so (2.6) is a codimension-one Riemann solution that is an intermediate boundary.

(E2) $\tilde{H}$ is independent of $U_n$, and $D_{U_0}\tilde{H}(U_0^*, U_n^*) \neq 0$. Thus (Q4) and ($Q6_2$) are satisfied, so (2.6) is a codimension-one Riemann solution that is a $U_L$-boundary.

Let us discuss the verificiation of (A) and (E1), which is necessary when $\tilde{H}$ depends on both $U_0$ and $U_n$. (The case in which $\tilde{H}$ is independent of $U_n$ is easier.) We first review some of the structure of structurally stable Riemann solutions.

Let (2.6) be a structurally stable Riemann solution. Let $m$ be an integer such that a one-wave or transitional wave group ends with $w_m$. Then a transitional or two-wave group begins with $w_{m+1}$.

Let $G_1(U_0, s_1, ..., s_m, U_m)$ and $G_2(U_m, s_{m+1}, ..., s_n, U_n)$ be the local defining maps for wave sequences of types $(T_1, ..., T_m)$ and $(T_{m+1}, ..., T_n)$ respectively, so that $G = (G_1, G_2)$. Then

(R1) $DG_1(U_0^*, s_1^*, ..., s_m^*, U_m^*)$, restricted to the space of vectors $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_m, \dot{U}_m)$ with $\dot{U}_0 = 0$, is surjective, with one-dimensional kernel spanned by a vector $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_m, \dot{U}_m)$ with $\dot{U}_m \neq 0$.

(R2) $DG_2(U_m^*, s_{m+1}^*, ..., s_n^*, U_n^*)$, restricted to the space of vectors $(\dot{U}_m, \dot{s}_{m+1}, ..., \dot{s}_n, \dot{U}_n)$ with $\dot{U}_n = 0$, is surjective, with one-dimensional kernel spanned by a vector $(\dot{U}_m, \dot{s}_{m+1}, ..., \dot{s}_n, \dot{U}_n)$ with $\dot{U}_m \neq 0$.

Therefore

(S1) There exist smooth mappings $s_i(U_0, \sigma)$ and $U_i(U_0, \sigma)$, $1 \leqslant i \leqslant m$, defined on $\mathscr{U}_0 \times (\sigma^* - \varepsilon, \sigma^* + \varepsilon)$, such that

$$s_i(U_0^*, \sigma^*) = s_i^* \qquad \text{and} \qquad U_i(U_0^*, \sigma^*) = U_i^*,$$

and for each $(U_0, \sigma)$,

$$U_0 \xrightarrow{s_1(U_0, \sigma)} \cdots \xrightarrow{s_m(U_0, \sigma)} (U_0, \sigma) \tag{4.3}$$

is an admissible wave sequence of type $(T_1, ..., T_m)$. Moreover, $(\partial U_m / \partial \sigma)(U_0^*, \sigma^*) \neq 0$.

(S2) There exist smooth mappings $\tilde{s}_i(U_n, \tau)$, $m+1 \leqslant i \leqslant n$, and $\tilde{U}_i(U_n, \tau)$, $m \leqslant i \leqslant n-1$, defined on $\mathscr{U}_n \times (\tau^* - \varepsilon, \tau^* - \varepsilon)$, such that

$$\tilde{s}_i(U_n^*, \tau^*) = s_i^* \qquad \text{and} \qquad \tilde{U}_i(U_n, \tau^*) = U_i^*,$$

and for each $(U_n, \tau)$,

$$\tilde{U}_m(U_n, \tau) \xrightarrow{\tilde{s}_{m+1}(U_n, \tau)} \cdots \xrightarrow{\tilde{s}_n(U_n, \tau)} U_n$$

is an admissible wave sequence of type $(T_{m+1}, ..., T_n)$. Moreover, $(\partial \tilde{U}_m/\partial \tau)(U_n^*, \tau^*) \neq 0$.

Of course, $(\partial U_m \partial \sigma)(U_0^*, \sigma^*)$ is a multiple of the vector $\dot{U}_m$ given by (R1), and $(\partial \tilde{U}_m/\partial \tau)(U_n^*, \tau^*)$ is a multiple of the vector $\dot{U}_m$ given by (R2). For a structurally stable Riemann solution, the wave group interaction condition implies that

(S3) $(\partial U_m/\partial \sigma)(U_0^*, \sigma^*)$ and $(\partial \tilde{U}_m/\partial \tau)(U_n^*, \tau^*)$ are linearly independent.

Then we have

PROPOSITION 4.1. *For each $(U_0, U_n)$ near $(U_0^*, U_n^*)$ there is a unique Riemann problem solution of type $(T_1, ..., T_n)$ near* (2.6).

*Proof.* The equation

$$U_m(U_0, \sigma) - \tilde{U}_m(U_n, \tau) = 0 \tag{4.4}$$

can be solved for $(\sigma, \tau)$ in terms of $(U_0, U_n)$ near $(U_0, \sigma, U_n, \tau) = (U_0^*, \sigma^*, U_n^*, \tau^*)$ by (S3) and the implicit function theorem. ∎

Alternatively, we can observe directly from (R1), (R2), and (S3) that $DG(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, restricted to the space of vectors $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, \dot{U}_n)$ with $\dot{U}_0 = 0$ and $\dot{U}_n = 0$, is surjective. In other words, (A) follows from (R1), (R2), and (S3).

Given a wave sequence that satisfies (MR), we shall verify (A) as follows. Let $w_j$ be the rarefaction of zero strength, and let $m \geqslant j$ be an integer such that a one-wave or transitional wave group ends with $w_m$. We shall check that (R1) holds. Statement (S1) then holds, except that (4.3) may not be admissible for every $(U_0, \sigma)$, because for some $(U_0, \sigma)$ we may have $s_j - \lambda_1(U_{j-1}) < 0$. Statements (R2), hence (S2), and (S3) follow from the statement of condition (MR). As observed above, from (R1), (R2), and (S3), it follows that (A) holds.

Recall that a map is *regular* at a point of its domain if its derivative there is surjective.

In the cases in which (E1) needs to be verified, $\sigma = s_j$, and $U_{j-1}$ is determined by $U_0$. Then from Eqs. (4.1) and (4.2) we have

$$\tilde{H}(U_0, U_n) = \sigma(U_0, U_n) - \lambda_1(U_{j-1}(U_0)), \tag{4.5}$$

where $\sigma(U_0, U_n) = s_j(U_0, U_n)$.

Proposition 4.2. *Let* (2.6) *satisfy* (MR) *and* (A). *Let* $\tilde{H}$ *be given by* (4.5). *Then* $D_{U_n}\tilde{H}(U_0^*, U_n^*)$ *is nonzero if and only if* $\tilde{U}_m(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

*Proof.* By (4.5), $D_{U_n}\tilde{H}(U_0^*, U_n^*) = D_{U_n}\sigma(U_0^*, U_n^*)$. To evaluate $D_{U_n}\sigma(U_0^*, U_n^*)$, in (4.4) we substitute $\sigma = \sigma(U_0, U_n)$ and $\tau = \tau(U_0, U_n)$, differentiate with respect to $U_n$, and set $(U_0, U_n, \sigma, \tau) = (U_0^*, U_n^*, \sigma^*, \tau^*)$. We obtain

$$\begin{aligned} &D_\sigma U_m(U_0^*, \sigma^*) \cdot D_{U_n}\sigma(U_0^*, U_n^*) \\ &\quad - D_{U_n}\tilde{U}_m(U_n^*, \tau^*) - D_\tau \tilde{U}_m(U_n^*, \tau^*) \cdot D_{U_n}\tau(U_0^*, U_n^*) = 0. \end{aligned} \tag{4.6}$$

Each summand is a $2 \times 2$ matrix. In the first summand, both columns are multiples of the column vector $D_\sigma U_m(U_0^*, \sigma^*)$; in the last summand, both columns are multiples of the column vector $D_\tau \tilde{U}_m(U_n^*, \tau^*)$. Since the column vectors $D_\sigma U_m(U_0^*, \sigma^*)$ and $D_\tau \tilde{U}_m(U_n^*, \tau^*)$ are linearly independent, $D_{U_n}\sigma(U_0^*, U_n^*)$ is nonzero if and only if the middle summand has a column independent of $D_\tau \tilde{U}_m(U_n^*, \tau^*)$, i.e., if and only if $\tilde{U}_m(U_n, \tau)$ is regular at $(U_n^*, \tau^*)$. ∎

Thus the verification of the second half of (E1), i.e., the verification that $D_{U_n}\tilde{H}(U_0^*, U_n^*)$ is nonzero, depends on properties of the wave sequence past the wave group of the missing rarefaction. For certain sequences of wave types $(T_{m+1}, ..., T_n)$, regularity of $\tilde{U}_m(U_n, \tau)$ follows from the hypotheses of the structural stability theorem; for others, it is an independent assumption, or it cannot hold. See Section 14.

To verify the first half of (E1), i.e., to verify that $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ is nonzero, we shall use the following proposition.

Proposition 4.3. *Let* (2.6) *satisfy* (MR) *and* (A). *Let* $\tilde{H}$ *be given by* (4.5) *with* $\sigma = s_j$. *Then* $D_{U_0}\tilde{H}(U_0^*, U_n^*)$ *is nonzero if and only if the equation*

$$DG_1(U_0^*, s_1^*, ..., s_m^*, U_m^*)(\dot{U}_0, \dot{s}_1, ..., \dot{s}_m, \dot{U}_m) = 0 \tag{4.7}$$

*has a solution* $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_m, \dot{U}_m)$ *such that*

(1) $\dot{U}_m$ *is a multiple of* $(\partial \tilde{U}_m/\partial \tau)(U_n^*, \tau^*)$.

(2) $\dot{s}_j - D\lambda_1(U_{j-1}^*)\,\dot{U}_{j-1} \neq 0$.

*Proof.* $D_{U_0}\widetilde{H}(U_0^*, U_n^*)\,\dot{U}_0$ is nonzero if and only if there is a vector $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, 0)$ such that

$$DG(U_0^*, s_1^*, ..., s_n^*, U_n^*)(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, 0) = 0 \tag{4.8}$$

and (2) holds. Now Eq. (4.8) is equivalent to Eq. (4.7) and the equation

$$DG_2(U_m^*, s_{m+1}^*, ..., s_n^*, U_n^*)(\dot{U}_m, \dot{s}_{m+1}, ..., \dot{s}_n, 0) = 0. \tag{4.9}$$

The solution set of Eq. (4.9) is one-dimensional, and each solution of Eq. (4.9) has $\dot{U}_m$ a multiple of $(\partial \widetilde{U}_m/\partial\tau)(U_n^*, \tau^*)$. Thus there is a vector $(\dot{U}_0, \dot{s}_1, ..., \dot{s}_n, 0)$ that satisfies Eq. (4.8) if and only if Eq. (4.7) has a solution that satisfies (1). ∎

This completes our discussion of Step 1 in the verification that a Riemann solution (2.6) satisfying (MR) is a codimension-one Riemann solution. We now turn our attention to Step 2.

Given a wave sequence (2.6) that satisfies (MR), there is a subsequence of one, two, or three waves consisting of the rarefaction of zero strength and any adjacent waves of the same speed. More precisely, there are integers $k < \ell$, with $\ell - k = 1$, 2, or 3, such that the wave sequence $(w_{k+1}, ..., w_\ell)$ includes the rarefaction of zero strength, the $* \cdot RS$ shock that immediately precedes it (if there is one), and the $RS \cdot *$ shock that immediately succeeds it (if there is one). In each case there is a naturally defined generalized shock $\tilde{w}_\ell$: $U_k^* \xrightarrow{s_\ell^*} U_\ell^*$ that can replace the subsequence $(w_{k+1}, ..., w_\ell)$. The new wave sequence

$$(w_1, ..., w_k, \tilde{w}_\ell, w_{\ell+1}, ..., w_n):$$

$$U_0^* \xrightarrow{s_1^*} \cdots \xrightarrow{s_k^*} U_k^* \xrightarrow{s_\ell^*} U_\ell^* \xrightarrow{s_{\ell+1}^*} \cdots \xrightarrow{s_n^*} U_n^* \tag{4.10}$$

is equivalent to (2.6). We remark:

(1) If $\ell - k = 1$, then the only difference between (4.10) and (2.6) is that a shock of zero strength has replaced a rarefaction of zero strength.

(2) If $\ell - k = 2$, then $\tilde{w}_\ell$ is actually a shock (the one that preceded or followed the rarefaction of zero strength in the original wave sequence).

(3) If $\ell - k = 3$, then $\tilde{w}_\ell$ is a generalized shock but not a shock. It is constructed by amalgamating the shocks that preceded and followed the rarefaction of zero strength in the original wave sequence.

We will check that there is a shock type $\hat{T}_\ell$ such that (4.10) is a codimension-one Riemann solution in the boundary of the structurally stable Riemann solutions of type $(T_1, ..., T_k, \hat{T}_\ell, T_{\ell+1}, ..., T_m)$. In order to do this, we will construct another local defining map $(G, H)$ for Riemann solutions of type $(T_1, ..., T_k, \hat{T}_\ell, T_{\ell+1}, ..., T_m)$ exhibiting a certain degeneracy in the wave of type $\hat{T}_\ell$. For this new map, we will verify (A), i.e., $(Q7_1)$, at the point $(U_0^*, s_1^*, ..., s_k^*, U_k^*, s_\ell^*, U_\ell^*, s_{\ell+1}^*, ..., s_n^*, U_n^*)$, and, if we verified (E$i$) for the map $(G, H)$ associated with (2.6), we will verify that the corresponding condition (E$i$) for the new map $(G, H)$. Thus (Q4) and $(Q6_2)$ hold for the new map; as in step 1, (Q1)–(Q3) and (Q5) follow from a geometric understanding of the situation.

This completes our discussion of Step 2. As to Step 3, in each case it follows from our construction in Step 2 that the two types of codimension-one Riemann solutions are defined on the same codimension-one surface $\mathscr{S}$ in $U_0 U_n F$-space, and that the two types of codimension-one Riemann solutions above a given point in $\mathscr{S}$ are equivalent. In the different cases we shall not discuss these facts; we shall, however, discuss the type of join that occurs. Here we only discuss deciding whether the join is regular or folded in case (E1) holds.

Let $\hat{\sigma}(U_0) = \lambda_1(U_{j-1}(U_0))$. Then for $U_0$ near $U_0^*$, and $\sigma$ near $\hat{\sigma}(U_0)$ with $\sigma \geqslant \hat{\sigma}(U_0)$, there are smooth mappings $s_i(U_0, \sigma)$ and $U_i(U_0, \sigma)$, $1 \leqslant i \leqslant m$, such that $s_i(U_0^*, \hat{\sigma}(U_0^*)) = s_i^*$, $U_i(U_0^*, \sigma(U_0^*) = U_i^*$, and for $\sigma > \hat{\sigma}(U_0)$, (4.3) is an admissible wave sequence of type $(T_1, ..., T_m)$. For $\sigma = \hat{\sigma}(U_0)$, one rarefaction has zero strength. (In fact, $\sigma = s_j$.)

Similarly, for $U_0$ near $U_0^*$, and $\sigma$ near $\hat{\sigma}(U_0)$ with $\sigma \leqslant \hat{\sigma}(U_0)$, there are smooth mappings $\hat{s}_i(U_0, \sigma)$ and $\hat{U}_i(U_0, \sigma)$, $1 \leqslant i \leqslant k$ and $\ell \leqslant i \leqslant m$, such that for each $(U_0, \sigma)$ with $\sigma < \hat{\sigma}(U_0)$,

$$
\begin{aligned}
U_0 &\xrightarrow{\hat{s}_i(U_0,\sigma)} \cdots \xrightarrow{\hat{s}_k(U_0,\sigma)} \hat{U}(U_0, \sigma) \xrightarrow{\hat{s}_l(U_0,\sigma)} \hat{U}_l(U_0, \sigma) \\
&\xrightarrow{\hat{s}_{l+1}(U_0,\sigma)} \cdots \xrightarrow{\hat{s}_m(U_0,\sigma)} \hat{U}_m(U_0, \sigma)
\end{aligned}
\tag{4.11}
$$

is an admissible wave sequence of type $(T_1, ..., T_k, \hat{T}_\ell, T_{\ell+1}, ..., T_m)$. For $\sigma = \hat{\sigma}(U_0)$, $\tilde{w}_\ell$ is a generalized shock.

We consider the forward wave curve $U_m(U_0^*, \sigma)$, $\sigma \geqslant \sigma^*$, and its continuation $\hat{U}_m(U_0^*, \sigma)$, $\sigma \leqslant \sigma^*$; and the backwards wave curve $\tilde{U}_m(U_n^*, \tau)$. We assume

(I) $(\partial U_m/\partial \sigma)(U_0^*, \sigma^*)$ and $(\partial \hat{U}_m/\partial \sigma)(U_0^*, \sigma^*)$ are both linearly independent of $(\partial \tilde{U}_m/\partial \tau)(U_n^*, \tau^*)$.

We distinguish two cases:

*Case* (Ia). $(\partial U_m/\partial \sigma)(U_0, \sigma(U_0))$ is a positive multiple of $(\partial \hat{U}_m/\partial \sigma)$ $(U_0, \sigma(U_0))$ for all $U_0$.

*Case* (Ib). Case (Ia) need not hold.

In Case (Ia) after reparametrization we can assume the multiple is identically one, so that $U_m$ and $\hat{U}_m$ fit together to form a $C^1$ mapping, which we again denote $U_m(U_0, \sigma)$. Then the proof of Proposition 4.7 allows us to solve for $\sigma$ and $\tau$ in terms of $(U_0, U_n)$ near $(U_0^*, U_n^*)$, which yields a unique Riemann solution near (2.6) for each $(U_0, U_n)$ near $(U_0^*, U_n^*)$. If $\sigma(U_0, U_n) < 0$, we have a structurally stable Riemann solution of one type; if $\sigma(U_0, U_n) > 0$, we have a structurally stable Riemann solution of another type.

PROPOSITION 4.4. *Let* (2.6) *satisfy* (MR). *Assume* (A) *and* (E1), *the analogues of* (A) *and* (E1) *discussed in Step* 2, *and* (Ia) *all hold. Then there is a codimension-one surface* $\mathscr{S}$ *in* $U_0U_n$*-space that separates regions in which different types of structurally stable Riemann solutions are defined. Thus the Riemann solution join is regular.*

*Proof.* From the proof of Proposition 4.9 we see that (E1) implies that $D_{U_n}\sigma(U_0^*, U_n^*) \neq 0$. The result follows. ∎

In Case (Ib) we distinguish two subcases.

*Case* (Ib)(i). $\det((\partial U_m/\partial\sigma)(U_0^*, \sigma^*), (\partial\tilde{U}_m/\partial\tau)(U_n^*, \tau^*))$ and $\det((\hat{U}_m/\partial\sigma)(U_0^*, \sigma^*), (\partial\tilde{U}_m/\partial\tau)(U_n^*, \tau^*))$ have the same sign. In this subcase the join is regular (see Fig. 4.1a).

*Case* (Ib)(ii). The determinants have opposite sign. In this subcase the join is folded (see Fig. 4.1b).

In Table III we indicate the nine missing rarefaction cases that are the subject of this paper by listing the waves preceding and following the rarefaction of zero strength *with the same wave speed.* Whether the missing

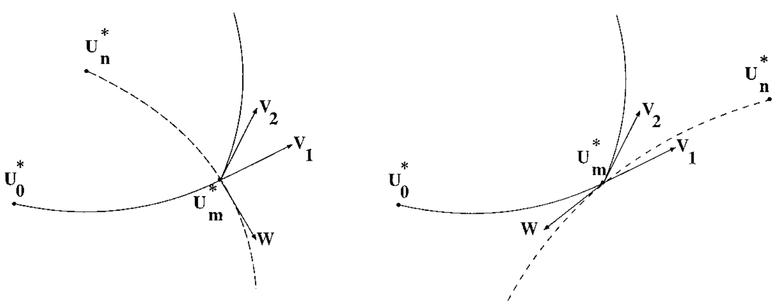


**FIG. 4.1.** Wave curves in cases in cases (Ib)(i) and (Ib)(ii). In each diagram, the curve $U_m(U_0^*, \sigma)$ and its continuation $\hat{U}_m(U_0^*, \sigma)$ are shown as solid curves; the curve $\tilde{U}_m(U_n^*, \tau)$ is dashed. The vectors $((\partial U_m/\partial\sigma)(U_0^*, \sigma^*)$, $(\hat{U}_m/\partial\sigma)(U_0^*, \sigma^*)$, and $(\partial\tilde{U}_m/\partial\tau)(U_n^*, \tau^*)$ are labeled $V_1$, $V_2$, and $W$, respectively, in the diagrams.

TABLE III

The Nine Classical Missing Rarefaction Cases

| Predecessor | Successor | Boundary type | Join type for first possibility |
|---|---|---|---|
| None | None | Intermediate or $U_L$ | Regular |
| None | $RS \cdot S$ | Intermediate or $U_L$ | Regular |
| None | $RS \cdot RS$ | $U_L$ | Regular |
| $R \cdot RS$ | None | Intermediate or $U_L$ | Regular |
| $R \cdot RS$ | $RS \cdot S$ | Intermediate or $U_L$ | Regular or folded |
| $R \cdot RS$ | $RS \cdot RS$ | $U_L$ | Regular or folded |
| $RS \cdot RS$ | None | Intermediate, $U_L$, $U_R$, or $F$ | Regular |
| $RS \cdot RS$ | $RS \cdot S$ | Intermediate, $U_L$, $U_R$ or $F$ | Regular or folded |
| $RS \cdot RS$ | $RS \cdot RS$ | $U_L$ or $F$ | Regular or folded |

rarefaction gives rise to an $F$-, $U_L$-, $U_R$-, or intermediate boundary depends on the location of the missing rarefaction in the wave sequence (see the end of Section 3); the table gives the possibilities. In each case, only the first boundary type listed is studied. Each case gives rise to a join, which in four cases may be folded; this information is also given in the table. In the remainder of the paper we state precisely the conditions under which these results hold and give proofs. However, the first case, a missing rarefaction with no preceding or following wave with the same wave speed, is stated without proof because of its familiarity.

## 5. NO PREDECESSOR, NO SUCCESSOR

THEOREM 5.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$, *with* $T_1 = R_1$, $T_2 \neq RS \cdot *$. *Assume*:

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_0^* \xrightarrow{s_1^*} U_1^*$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_1(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot S, T_2, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a regular join that is an intermediate boundary.*

We shall not prove this result, since its essence, which is that the slow rarefaction and slow $(R \cdot S)$ shock curves emanating from the left state fit together to form a smooth curve, is so familiar; see Ref. [5]. The interested reader may construct a proof modeled on those in the other sections.

## 6. NO PREDECESSOR, SUCCESSOR $RS \cdot S$

THEOREM 6.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$, *with* $T_1 = R_1$, $T_2 = RS \cdot S$. *Assume*:

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_0^* \xrightarrow{s_1^*} U_1^*$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_2(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot S, T_3, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a regular join that is an intermediate boundary.*

*Proof.* *Step* 1. The one-wave group of (2.6) is

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^*.$$

We have

$$s_1^* = s_2^* = \lambda_1(U_1^*) \qquad \text{and} \qquad U_0^* = U_1^*. \tag{6.1}$$

We note that $(U_0, s_1, U_1, s_2, U_2)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ represents an admissible wave sequence of type $(R_1, RS \cdot S)$ if and only if

$$U_1 - \psi(U_0, s_1) = 0, \tag{6.2}$$

$$s_1 - \lambda_1(U_0) \geqslant 0, \tag{6.3}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{6.4}$$

$$\lambda_1(U_1) - s_2 = 0. \tag{6.5}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot S, T_3, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s_2, U_2)$ is given by the left hand sides of Eqs. (6.2) and (6.4)–(6.5), and $G_2(U_2, s_3, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(T_3, ..., T_n)$. The linearization of Eqs. (6.2) and (6.4)–(6.5) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ is

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{6.6}$$

$$(DF(U_2^*) - s_2^* I)\, \dot{U}_2 - (DF(U_1^*) - s_1^* I)\, \dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{6.7}$$

$$D\lambda_1(U_1^*)\, \dot{U}_1 - \dot{s}_2 = 0. \tag{6.8}$$

Solutions of Eqs. (6.6)–(6.8) with $\dot{U}_0=0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2)=(0, 1, r_1(U_1^*), 1, (DF(U_2^*)-s_1^* I)^{-1}(U_2^*-U_1^*)). \tag{6.9}$$

Thus (R1) holds. Since (R2) and (S3) follow from assumption (1), (A) holds.

Solutions of (6.2)–(6.3) near $(U_0^*, s_1^*, U_2^*, s_2^*, U_2^*)$ are parameterized by $U_0$ and $s_1$ as

$$U_1=\psi(U_0, s_1), \qquad s_1 \geqslant \lambda_1(U_0), \tag{6.10}$$

$$s_2=\lambda_1(U_1), \tag{6.11}$$

$$U_2=\varphi(U_1, \lambda_1(U_1)). \tag{6.12}$$

Here (6.12) is the solution of Eq. (6.4), given near $(U_1^*, s_2^*, U_2^*)$ by the implicit function theorem.

The left-hand side of (6.3) is the map $H$ for this situation, so $\tilde{H}(U_0, U_n) = s_1(U_0, U_n)-\lambda_1(U_0)$. The second part of (E1) holds by Proposition 4.2.

To verify the first part of (E1) we note that Eqs. (6.6)–(6.8) have the solution $(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2)=(r_1(U_0^*), 0, 0, 0, 0)$. Thus the hypotheses of Proposition 4.3 are satisfied with $j=1$ and $m=2$, so the first part of (E1) holds.

*Step* 2. The bifurcation diagram of $\dot{U}=F(U)-F(U_0^*)-s(U-U_0^*)$ features (1) a transcritical bifurcation at $(U, s)=(U_0^*, \lambda_1(U_0^*))$, and (2) at $s=\lambda_1(U_0^*)$, a nondegenerate repeller-saddle to saddle connection from $U_0^*$ to $U_2^*$. These features persist for $U_0$ near $U_0^*$; see Fig. 6.1. We conclude from this figure that if $(U_0, s, U)$ near $(U_0^*, s_2^*, U_2^*)$ represents a shock of type $R \cdot S$ or $RS \cdot S$, then

$$F(U)-F(U_0)-s(U-U_0)=0, \tag{6.13}$$

$$\lambda_1(U_0)-s \geqslant 0. \tag{6.14}$$

Solutions of Eq. (6.13) with $\lambda_1(U_0)-s>0$ represent $R \cdot S$ shocks; those with $\lambda_1(U_0)-s=0$ represent $RS \cdot S$ shocks.

Let $G(U_0, s, U, s_3, U_3, s_4, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot S, T_3, ..., T_n)$ near $(U_0^*, s_2^*, U_2^*, s_3^*, U_3^*, s_4^*, ..., s_n^*, U_n^*)$, $G=(G_1, G_2)$, where $G_1(U_0, s, U)$ is given by the left-hand side of Eq. (6.13), and $G_2(U, s_3, U_3, s_4, ..., s_n, U_n)$ is as in Step 1. The linearization of Eq. (6.13) at $(U_0^*, s_2^*, U_2^*)$ is

$$(DF(U_2^*)-s_2^* I)\dot{U}-(DF(U_0^*)-s_2^* I)\dot{U}_0-\dot{s}(U_2^*-U_0^*)=0. \tag{6.15}$$

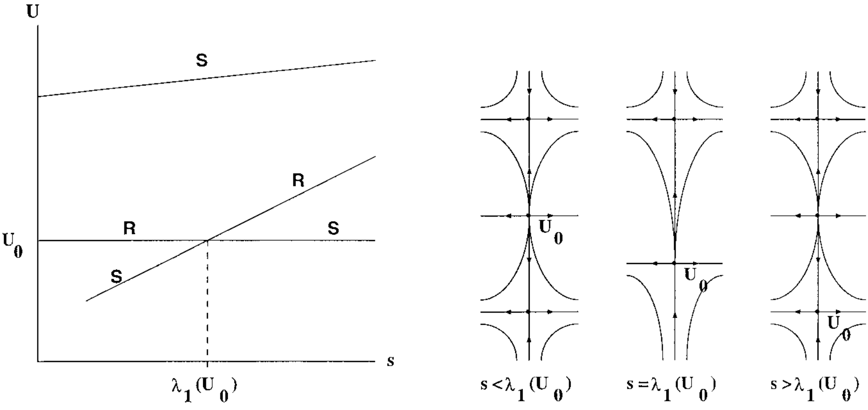


**FIG. 6.1.** Bifurcation diagram and phase portraits for $\dot{U}=F(U)-F(U_0)-s(U-U_0)$, $U_0$ fixed near $U_0^*$.

Solutions of Eq. (6.15) with $\dot{U}_0=0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}, \dot{U})=(0, 1, (DF(U_2^*)-s_2^* I)^{-1}(U_2^*-U_0^*). \tag{6.16}$$

Therefore (R1) holds. Statement (R2) holds as before. Since the last component of (6.9) agrees with the last component of (6.16), (S3) holds. Therefore (A) holds.

Solutions of (6.13)–(6.14) near $(U_0^*, s_2^*, U_2^*)$ are parameterized by $U_0$ and $s \leqslant \lambda_1(U_0)$,

$$U=U(U_0, s), \qquad s \leqslant \lambda_1(U_0), \tag{6.17}$$

where $U(U_0, s)$ is the solution of (6.13) given near $(U_0^*, s_2^*, U_2^*)$ by the implicit function theorem.

The map $H$ for this situation is $\lambda_1(U_0)-s$, so $\tilde{H}(U_0, U_n)=\lambda_1(U_0)-s(U_0, U_n)$. The second part of (E1) holds by the proof of Proposition 4.3.

To verify the first part of (E1), we note that one solution of (6.15) is $(\dot{U}_0, \dot{s}, \dot{U})=(r_1(U_0^*), 0, 0)$ and use the proof of Proposition 4.3.

*Step* 3. We note that $U_2(U_0, s_1)$ is defined for $s_1 \geqslant \lambda_1(U_0)$, $U(U_0, s)$ is defined for $s \leqslant \lambda_1(U_0)$, and $(\partial U_2/\partial s_1)(U_0, \lambda_1(U_0))=(\partial U/\partial s)(U_0, \lambda_1(U_0))$. (This equation holds at $U_0^*$ by (6.9) and (6.16), and it holds nearby by the same argument.) Therefore (Ia) holds, so the join is regular. ∎

*Remark.* The observation that (Ia) holds is equivalent to the observation that when repeller-to-saddle shocks become degenerate at the repeller end, the one-wave curve continues smoothly as $(R_1, RS \cdot S)$ composites. This observation can be verified under less restrictive assumptions by the methods of Ref. [12].

## 7. NO PREDECESSOR, SUCCESSOR $RS \cdot RS$

THEOREM 7.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$, *with* $T_1 = R_1$, $T_2 = RS \cdot RS$, $T_3 = R_1$. *Assume all hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_0^* \xrightarrow{s_1^*} U_1^*$ *has zero strength. Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot RS, R_1, T_4, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a regular join that is a* $U_L$-*boundary.*

*Proof.* *Step* 1. The one-wave group of (2.6) begins

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^*$$

(and may be longer). We have

$$s_1^* = s_2^* = \lambda_1(U_1^*) = \lambda_1(U_2^*) \qquad \text{and} \qquad U_0^* = U_1^*.$$

We note that $(U_0, s_1, U_1, s_2, U_2)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ represents an admissible wave sequence of type $(R_1, RS \cdot RS)$ if and only if

$$U_1 - \psi(U_0, s_1) = 0, \tag{7.1}$$

$$s_1 - \lambda_1(U_0) \geqslant 0, \tag{7.2}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{7.3}$$

$$\lambda_1(U_1) - s_2 = 0, \tag{7.4}$$

$$\lambda_1(U_2) - s_2 = 0. \tag{7.5}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot RS, R_1, T_4, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s_2, U_2)$ is given by the left hand sides of Eqs. (7.1) and (7.3)–(7.5), and $G_2(U_2, s_3, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_4, ..., T_n)$. From the theory of [9],

$$DG_1(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*), \text{ restricted to}$$
$$\{(\dot U_0, \dot s_1, \dot U_1, \dot s_2, \dot U_2) : \dot U_0 = 0\}, \text{ is an isomorphism,} \tag{7.6}$$

and

$$DG_2(U_2^*, s_3^*, ..., s_n^*, U_n^*), \text{ restricted to}$$
$$\{(\dot{U}_2, \dot{s}_3, ..., \dot{s}_n, \dot{U}_n) : \dot{U}_2 = \dot{U}_n = 0\}, \text{ is an isomorphism.} \quad (7.7)$$

(For (7.7), note that the wave sequence $(w_3^*, ..., w_n^*)$ satisfies the hypotheses of Theorem 2.4.) Therefore (A) holds.

From (7.6), we can solve Eqs. (7.1) and (7.3)–(7.5) for $(s_1, U_1, s_2, U_2)$ in terms of $U_0$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$. A solution of $G=0$ represents a Riemann solution of the desired type if and only if $s_1 - \lambda_1(U_0) \geqslant 0$. By the definition of $\psi$, $s_1 = \lambda_1(U_1)$, and $\lambda_1(U_1) = s_2$ by Eq. (7.4). Therefore we study the function $\tilde{H}(U_0) := s_2(U_0) - \lambda_1(U_0)$. Condition (E2) will be verified if we show that $D\tilde{H}(U_0^*) \neq 0$. We will calculate $D\tilde{H}(U_0^*)\dot{U}_0$ by linearizing Eqs. (7.1) and (7.3)–(7.5) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$, solving for $\dot{s}_2$ in terms of $\dot{U}_0$, and then subtracting $D\lambda_1(U_0^*)\dot{U}_0$.

Actually, we need only linearize Eqs. (7.1) and (7.3) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$,

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \quad (7.8)$$

$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_2^* I)\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0. \quad (7.9)$$

In Eqs. (7.8) and (7.9) we write

$$\dot{U}_0 = ar_1(U_0^*) + br_2(U_0^*),$$
$$\dot{U}_1 = cr_1(U_1^*) + dr_2(U_1^*).$$

We multiply Eq. (7.8) by $\ell_2(U_0^*)$ and Eq. (7.9) by $\ell_1(U_2^*)$. We get (using Lemma 2.2 on Eq. (7.8))

$$d - b = 0,$$
$$-d(\lambda_2(U_1^*) - s_2^*)\,\ell_1(U_2^*)\,r_2(U_1^*) - \dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0.$$

Therefore

$$\begin{aligned} D\tilde{H}(U_0^*)\dot{U}_0 &= \dot{s}_2 - D\lambda_1(U_0^*)\dot{U}_0 = \dot{s}_2 - (a + b\, D\lambda_1(U_0^*)\,r_2(U_0^*)) \\ &= -a - b\left\{ D\lambda_1(U_0^*)\,r_2(U_0^*) \right. \\ &\quad \left. + \frac{(\lambda_2(U_1^*) - \lambda_1(U_1^*))\,\ell_1(U_2^*)\,r_2(U_1^*)}{\ell_1(U_2^*)(U_2^* - U_1^*)} \right\}. \end{aligned} \quad (7.10)$$

Clearly $D\tilde{H}(U_0^*) \neq 0$.

Therefore

$$\mathscr{C} = \{ U_0 : \widetilde{H}(U_0) = 0\}$$

is a smooth curve near $U_0^*$; it is transverse $r_1(U_0^*)$; and for $(U_0, U_n)$ near $(U_0^*, U_n^*)$, a solution of type $(R_1, RS \cdot RS, R_1, T_4, ..., T_n)$ exists provided $U_0$ is on the side of $\mathscr{C}$ opposite that to which $r_1(U_1^*)$ points.

*Step* 2. We consider the point $(U_0^*, s_2^*, U_2^*, s_3^*, U_3^*, s_4^*, ..., s_n^*, U_n^*)$ in $\mathbb{R}^{3n-1}$. We shall investigate the existence of nearby points $(U_0, s, U, s_3, U_3, s_4, ..., s_n, U_n)$ that represent Riemann solutions of type $(R \cdot RS, R_1, T_4, ..., T_n)$. To obtain a condition for the existence of such points, we consider the bifurcation diagram of

$$\dot{U} = F(U) - F(U_0) - s(U - U_0).$$

For $U_0 = U_0^*$, it is shown in Fig. 7.1a. (The parabolic curve through $U_2^*$ may open to the left, but this changes nothing.) For $U_0$ near $U_0^*$, the bifurcation diagram of Fig. 7.1a can perturb to one of those shown in Fig. 7.1b or 7.1c. There is an $R \cdot RS$ shock from $U_0$ to $U(U_0)$ if and only if the bifurcation diagram is as in Fig. 6.1b. For the bifurcation diagram of Fig. 7.1a, there is a generalized shock from $U_0$ to $U(U_0)$.

To study this situation, we consider the system

$$F(U) - F(U_0) - s(U - U_0) = 0, \tag{7.11}$$

$$\lambda_1(U) - s = 0, \tag{7.12}$$

near $(U_0, s, U) = (U_0^*, s_2^*, U_2^*)$. Let $G(U_0, s, U, s_3, U_3, s_4, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot RS, R_1, T_4, ..., T_n)$ near $(U_0^*, s_2^*, U_2^*, s_3^*, U_3^*, s_4^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s, U)$ is given by the left-hand sides of Eqs. (7.11)–(7.12), and $G_2(U, s_3, U_3, s_4, ..., s_n, U_n)$ is as in Step 1. From the proof of Lemma 5.3 in Ref. [9], $DG_1(U_0^*, s_2^*, U_2^*)$, restricted to $\{(\dot{U}_0, \dot{s}, \dot{U}) : \dot{U}_0 = 0\}$, is an isomorphism. Then from (7.7), (A) holds. Moreover, Eqs. (7.11)–(7.12), can be solved for $(s, U)$ in terms of $U_0$ near $U_0^*$; the solution is $(s(U_0), U(U_0))$.

From Fig. 7.1, a solution of $G = 0$ represents a wave sequence of the desired type if and only if $s(U_0) < \lambda_1(U_0)$. We therefore study the function $\widetilde{H}(U_0) := \lambda_1(U_0) - s(U_0)$.

To verify (E2), we calculate $D\widetilde{H}(U_0^*)\dot{U}_0$.

Linearizing Eq. (7.11) at $(U_0^*, s_2^*, U_2^*)$ yields

$$(DF(U_2^*) - s_2^* I)\dot{U} - (DF(U_0^*) - s_2^* I)\dot{U}_0 - \dot{s}(U_2^* - U_0^*) = 0.$$

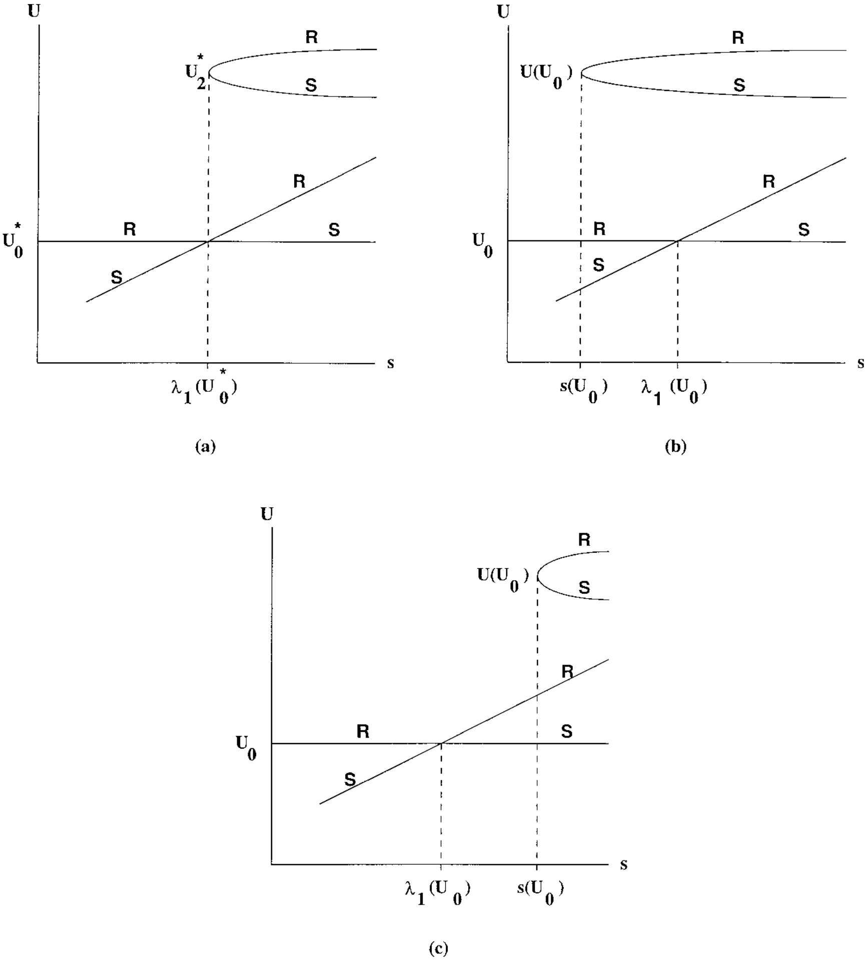


**FIG. 7.1.** Bifurcation diagrams for $\dot{U} = F(U) - F(U_0) - s(U - U_0)$, $U_0$ fixed near $U_0^*$. The proof shows that bifurcation diagram (a) actually occurs for any $U_0$ on the curve $\mathscr{C}$ defined in the proof; the other two diagrams occur for $U_0$ on opposite sides of $\mathscr{C}$.

Multiplying by $\ell_1(U_2^*)$ yields

$$-b(\lambda_2(U_0^*) - s_2^*)\,\ell_1(U_2^*)\,r_2(U_0^*) - \dot{s}\ell_1(U_2^*)(U_2^* - U_0^*) = 0.$$

Therefore

$$D\tilde{H}(U_0^*)\,\dot{U}_0 = D\lambda_1(U_0^*)\,\dot{U}_0 - \dot{s} = -(\text{right hand side of Eq. (7.10)}).$$

Thus $D\tilde{H}(U_0^*) \neq 0$, so (E1) holds.

It is easy to see that $\{U_0: \tilde{H}(U_0)=0\}$, where $\tilde{H}$ comes from Step 2, is precisely the curve $\mathscr{C}$ defined in Step 1. We remark that for $U_0$ on $\mathscr{C}$, we have the bifurcation diagram of Fig. 7.1a; for $U_0$ on the side of $\mathscr{C}$ where $\tilde{H}>0$ (resp. $\tilde{H}<0$), we have the bifurcation diagram of Fig. 7.1b (resp. Fig. 7.1c).

*Step* 3. Our two functions $\tilde{H}$ defined in Steps 1 and 2 are positive on opposite sides of $\mathscr{C}$, so the Riemann solution join is regular. ▮

## 8. PREDECESSOR $R \cdot RS$, NO SUCCESSOR

THEOREM 8.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$ *with* $T_1 = R \cdot RS$, $T_2 = R_1$, $T_3 \neq RS \cdot *$. *Assume*

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_1^* \xrightarrow{s_2^*} U_2^*$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_2(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot S, T_3, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a regular join that is an intermediate boundary.*

*Proof.* *Step* 1. The one-wave group of (2.6) is

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^*.$$

We have

$$s_1^* = s_2^* = \lambda_1(U_1^*) \qquad \text{and} \qquad U_1^* = U_2^*.$$

We note that $(U_0, s_1, U_1, s_2, U_2)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ represents an admissible wave sequence of type $(R \cdot RS, R_1)$ if and only if

$$F(U_1) - F(U_0) - s_1(U_1 - U_0) = 0, \tag{8.1}$$

$$\lambda_1(U_1) - s_1 = 0, \tag{8.2}$$

$$U_2 - \psi(U_1, s_2) = 0, \tag{8.3}$$

$$s_2 - \lambda_1(U_1) \geqslant 0. \tag{8.4}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot RS, R_1, T_3, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s_2, U_2)$ is given by the left hand sides of Eqs. (8.1)–(8.3), and $G_2(U_2, s_3, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(T_3, ..., T_n)$. The linearization of Eqs. (8.1)–(8.3) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ is

$$(DF(U_1^*) - s_1^* I)\,\dot{U}_1 - (DF(U_0^*) - s_1^* I)\,\dot{U}_0 - \dot{s}_1(U_1^* - U_0^*) = 0, \tag{8.5}$$

$$D\lambda_1(U_1^*)\,\dot{U}_1 - \dot{s}_1 = 0, \tag{8.6}$$

$$\dot{U}_2 - D\psi(U_1^*, s_2^*)(\dot{U}_1, \dot{s}_2) = 0. \tag{8.7}$$

Solutions of Eqs. (8.5)–(8.7) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2) = (0, 0, 0, 1, r_1(U_2^*)). \tag{8.8}$$

Thus (R1) holds. Since (R2) and (S3) follow from the assumptions of the theorem, (A) holds.

Solutions of (8.1)–(8.4) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ are parametrized by $U_0$ and $s_2$ as

$$s_1 = s_1(U_0), \tag{8.9}$$

$$U_1 = U_1(U_0), \tag{8.10}$$

$$U_2 = \psi(U_1, s_2), \qquad s_2 \geqslant \lambda_1(U_1). \tag{8.11}$$

Here (8.9)–(8.10) is the solution of (8.1)–(8.2), and $s_1(U_0) = \lambda_1(U_1(U_0))$.

The solution (8.9)–(8.10) may be understand by considering the bifurcation diagram of

$$\dot{U} = F(U) - F(U_0) - s(U - U_0)$$

for fixed $U_0$ near $U_0^*$; see Fig. 8.1. (The parabolic curve may open to the left, but this changes nothing.) In this bifurcation diagram there is a saddle-node bifurcation at $s = s_1(U_0)$, at the point $U = U_1(U_0)$.

The left-hand side of (8.4) is the map $H$ for this situation, so $\tilde{H}(U_0, U_n) = s_2(U_0, U_n) - \lambda_1(U_1(U_0))$. The second part of (E1) holds by Proposition 4.2. To verify the first part of (E1), we note that Eqs. (8.5)–(8.7) have the solution

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2) = (-(DF(U_0^*) - s_1^* I)^{-1}(U_1^* - U_0^*), 1, r_1(U_1^*), 0, 0). \tag{8.12}$$

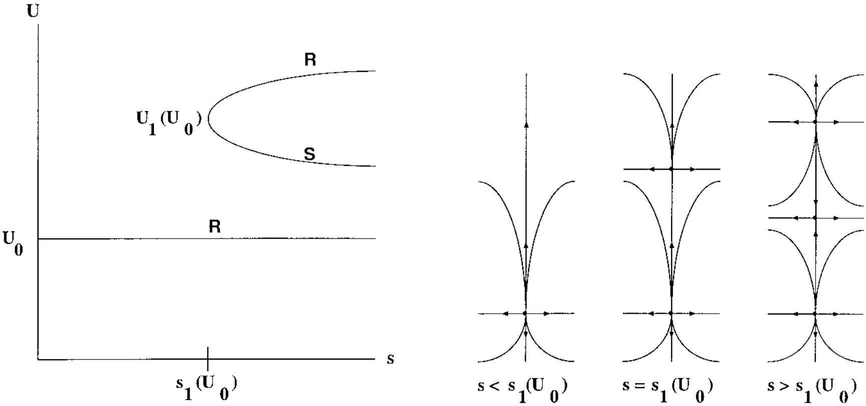


**FIG. 8.1.** Bifurcation diagram and phase portraits for $\dot{U}=F(U)-F(U_0)-s(U-U_0)$, $U_0$ fixed near $U_0^*$.

Thus the hypotheses of Proposition 4.3 are satisfied with $j=2$ and $m=2$, so the first part of (E1) holds.

*Step* 2. We note that if $(U_0, s, U)$ near $(U_0^*, s_1^*, U_1^*)$ represents a shock of type $R \cdot S$ or $R \cdot RS$, then

$$F(U)-F(U_0)-s(U-U_0)=0, \tag{8.13}$$

$$\lambda_1(U)-s \leqslant 0. \tag{8.14}$$

The inequality (8.14) simply says that an eigenvalue of $\dot{U}=F(U)-F(U_0)-s(U-U_0)$ at $U$ is non-positive. Solutions of (8.13) with $\lambda_1(U)-s<0$ represent $R \cdot S$ shocks; those with $\lambda_1(U)-s=0$ represent $R \cdot RS$ shocks.

Let $G(U_0, s, U, s_3, U_3, s_4, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot S, T_3, ..., T_n)$ near $(U_0^*, s_1^*, U_1^*, s_3^*, U_3^*, s_4^*, ..., s_n^*, U_n^*)$, $G=(G_1, G_2)$, where $G_1(U_0, s, U)$ is given by the left-hand side of Eq. (8.13), and $G_2(U, s_3, U_3, s_4, ..., s_n, U_n)$ is as in Step 1. The linearization of Eq. (8.13) at $(U_0^*, s_1^*, U_1^*)$ is

$$(DF(U_1^*)-s_1^* I)\,\dot{U}-(DF(U_0^*)-s_1^* I)\,\dot{U}_0-\dot{s}(U_1^*-U_0^*)=0. \tag{8.15}$$

Solutions of (8.15) with $\dot{U}_0=0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}, \dot{U})=(0, 0, r_1(U_1^*)). \tag{8.16}$$

Therefore (R1) holds. Since the last component of (8.8) agrees with the last component of (8.16), (A) holds.

Solutions of Eq. (8.13) near $(U_0^*, s_1^*, U_1^*)$ can by parameterized by $U_0$ and a parameter $t$ near 0 as

$$s = s(U_0, t), \tag{8.17}$$

$$U = U(U_0, t). \tag{8.18}$$

We can easily arrange that

$$s(U_0, 0) = s_1(U_0), \tag{8.19}$$

$$\frac{\partial s}{\partial t}(U_0, 0) = 0, \tag{8.20}$$

$$U(U_0, 0) = U_1(U_0), \tag{8.21}$$

$$\frac{\partial U}{\partial t}(U_0, 0) = r_1(U_1(U_0)). \tag{8.22}$$

The map $H$ for this situation is $H(U_0, s, U, s_3, ..., s_n, U_n) = s - \lambda_1(U)$. Therefore

$$\tilde{H}(U_0, U_n) = s(U_0, t(U_0, U_n)) - \lambda_1(U(U_0, t(U_0, U_n))). \tag{8.23}$$

Differentiating with respect to $U_n$ yields

$$\begin{aligned} D_{U_n}\tilde{H}(U_0^*, U_n^*, F^*) = {} & \frac{\partial s}{\partial t}(U_0^*, 0)\, D_{U_n} t(U_0^*, U_n^*) \\ & - D\lambda_1(U_1^*)\, D_t U(U_0^*, s_1^*)\, D_{U_n} t(U_0^*, U_n^*). \end{aligned} \tag{8.24}$$

By (8.20), (8.22), and (2.3), (8.24) simplifies to

$$\begin{aligned} D_{U_n}\tilde{H}(U_0^*, U_n^*, F^*) &= -D\lambda_1(U_1^*)\, r_1(U_1^*)\, D_{U_n} t(U_0^*, U_n^*) \\ &= -D_{U_n} t(U_0^*, U_n^*). \end{aligned}$$

The proof of Proposition 4.2 shows that this is not zero, so the second half of (E1) is verified.

To verify the first half of (E1), we note that one solution of Eq. (8.15) is $(\dot{U}_0, \dot{s}, \dot{U}) = (-DF(U_0^* - s_1^* I)^{-1}(U_1^* - U_0^*), 1, 0)$. But then

$$D_{U_0}\tilde{H}(U_0^*, U_n^*)\, \dot{U}_0 = \dot{s} - D\lambda_1(U_1^*)\, \dot{U} = 1 - 0 = 1.$$

*Step* 3. We note that in Step 1,

$$U_2 = \psi(U_1(U_0), s_2)$$

is defined for $s_2 \geqslant \lambda_1(U_1(U_0))$, while in Step 2, $U(U_0, t)$ satisfies the inequality (8.14) for $t \leqslant 0$. The latter fact follows from the calculation

$$D\lambda_1(U(U_0^*, 0))\, D_t U(U_0^*, 0) - D_t s(U_0^*, 0) = D\lambda_1(U_1^*)\, r_1(U_1^*) - 0 = 1.$$

Then from (8.8) and (8.16), (Ia) holds, so the join is regular. ∎

*Remark.* The observation that (Ia) holds is equivalent to the observation that when repeller-to-saddle shocks become degenerate at the saddle end, the one-wave curve continues smoothly as $(R \cdot RS, R_1)$ composites. This observation is verified under less restrictive assumptions in Ref. [12].

## 9. PREDECESSOR $R \cdot RS$, SUCCESSOR $RS \cdot S$

THEOREM 9.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$ *with* $T_1 = R \cdot RS$, $T_2 = R_1$, $T_3 = RS \cdot S$. *Assume*

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_1^* \xrightarrow{s_2^*} U_2^*$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_3(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

(3) $(DF(U_3^*) - s_1^* I)^{-1}(U_3^* - U_0^*)$ *and* $(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$ *are linearly independent.*

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot S, T_4, ..., T_n)$. *Riemann solution* (2.6) *(and its equivalent) lies in a join that is an intermediate boundary. The join may be regular or folded.*

*Proof. Step* 1. The one-wave group of (2.6) is

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^*.$$

We have

$$s_1^* = s_2^* = s_3^* = \lambda_1(U_1^*) \qquad \text{and} \qquad U_1^* = U_2^*. \tag{9.1}$$

We note that $(U_0, s_1, ..., s_3, U_3)$ near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ represents an admissible wave sequence of type $(R \cdot RS, R_1, RS \cdot S)$ if and only if

$$F(U_1)-F(U_0)-s_1(U_1-U_0)=0, \tag{9.2}$$

$$\lambda(U_1)-s_1=0, \tag{9.3}$$

$$U_2-\psi(U_1, s_2)=0, \tag{9.4}$$

$$s_2-\lambda_1(U_1)\geqslant 0, \tag{9.5}$$

$$F(U_3)-F(U_2)-s_3(U_3-U_2)=0, \tag{9.6}$$

$$\lambda_1(U_2)-s_3=0. \tag{9.7}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot RS, R_1, RS \cdot S, T_4, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G=(G_1, G_2)$, where $G_1(U_0, s_1, U_1, s_2, U_2, s_3, U_3)$ is given by the left hand sides of Eqs. (9.2)–(9.4) and (9.6)–(9.7), and $G_2(U_3, s_4, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(T_4, ..., T_n)$. The linearization of Eqs. (9.2)–(9.4) and (9.6)–(9.7) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*, s_3^*, U_3^*)$ is

$$(DF(U_1^*)-s_1^* I)\dot{U}_1-(DF(U_0^*)-s_1^* I)\dot{U}_0-\dot{s}_1(U_1^*-U_0^*)=0, \tag{9.8}$$

$$D\lambda_1(U_1^*)\dot{U}_1-\dot{s}_1=0, \tag{9.9}$$

$$\dot{U}_2-D\psi(U_1^*, s_2^*)(\dot{U}_1, \dot{s}_2)=0, \tag{9.10}$$

$$(DF(U_3^*)-s_3^* I)\dot{U}_3-(DF(U_2^*)-s_3^* I)\dot{U}_2-\dot{s}_3(U_3^*-U_2^*)=0, \tag{9.11}$$

$$D\lambda_1(U_2^*)\dot{U}_2-\dot{s}_3=0. \tag{9.12}$$

Solutions of (9.8)–(9.12) with $\dot{U}_0=0$ form a one-dimensional space spanned by

$$\begin{aligned}(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3)&\\ =(0, 0, 0, 1, r_1(U_2^*), 1, (DF(U_3^*)-s_3^* I)^{-1}(U_3^*-U_2^*).&\end{aligned} \tag{9.13}$$

Thus (R1) holds. Since (R2) and (S3) follow from assumtion (1), (A) holds.

Solutions of (9.2)–(9.7) near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ are parameterized by $U_0$ and $s_2$ as

$$s_1=s_1(U_0), \tag{9.14}$$

$$U_1=U_1(U_0), \tag{9.15}$$

$$U_2=\psi(U_1, s_2), \qquad s_2\geqslant s_1(U_0), \tag{9.16}$$

$$s_3=s_2, \tag{9.17}$$

$$U_3=\varphi(U_2, s_3). \tag{9.18}$$

Here $(s_1(U_0), U_1(U_0))$ is the solution of (9.2)–(9.3) near $(U_0^*, s_1^*, U_1^*)$, where $s_1(U_0) = \lambda_1(U_1(U_0))$, and $U_3 = \varphi(U_2, s_3)$ is the solution of (9.6) near $(U_2^*, s_3^*, U_3^*)$.

The left-hand side of (9.5) is the map $H$ for this situation, so $\tilde{H}(U_0, U_n) = s_2(U_0, U_n) - \lambda_1(U_1(U_0))$. The second part of (E1) holds by Proposition 4.2. To verify the first part of (E1), we note that Eqs. (9.8)–(9.12) have the solution

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3)$$
$$= (-(DF(U_0^*) - s_1^* I)^{-1} (U_1^* - U_0^*), 1, r_1(U_1^*), 0, 0, 0, 0). \qquad (9.19)$$

Thus the hypotheses of Proposition 4.3 are satisfied with $j = 2$ and $m = 3$, so the first part of (E1) holds.

*Step* 2. We look for points $(U_0, s, U)$ near $(U_0^*, s_1^*, U_3^*)$ that represent $R \cdot S$ shocks. To see that such points exist, we consider the bifurcation diagram of

$$\dot{U} = F(U) - F(U_0) - s(U - U_0)$$

for $U_0$ near $U_0^*$. It is given by Fig. 9.1 in the case $\ell_1(U_1^*)(U_1^* - U_0^*) > 0$; if $\ell_1(U_1^*)(U_1^* - U_0^*) < 0$, the parabolic curve through $U_1(U_0)$ opens to the left. (We have $\ell_1(U_1^*)(U_1^* - U_0^*) \neq 0$ by the nondegeneracy conditions for waves of type $R \cdot RS$.)

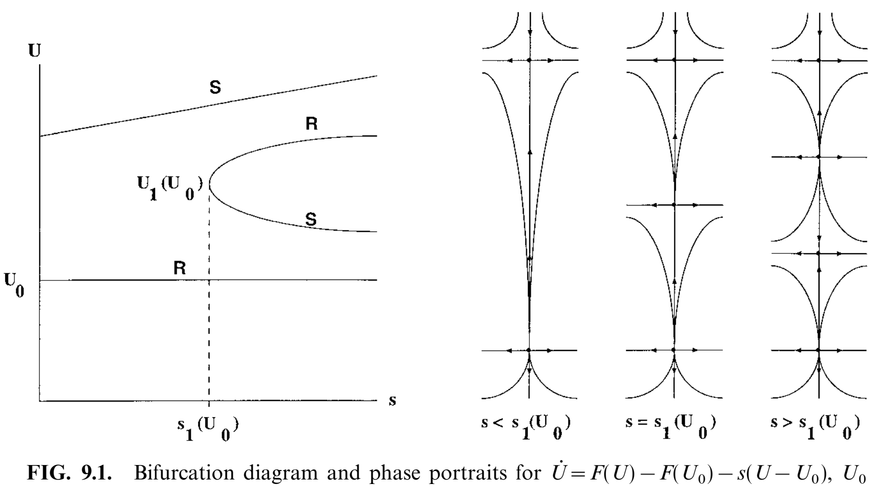


**FIG. 9.1.** Bifurcation diagram and phase portraits for $\dot{U} = F(U) - F(U_0) - s(U - U_0)$, $U_0$ fixed near $U_0^*$.

In the case $\ell_1(U_1^*)(U_1^* - U_0^*) > 0$, if $(U_0, s, U)$ near $(U_0^*, s_1^*, U_3^*)$ represents an $R \cdot S$ shock, then from Fig. 9.1,

$$F(U) - F(U_0) - s(U - U_0) = 0, \tag{9.20}$$

$$s_1(U_0) - s \geqslant 0. \tag{9.21}$$

In fact, solutions of Eq. (9.20) with $s_1(U_0) - s > 0$ represent $R \cdot S$ shocks; those with $s_1(U_0) - s = 0$ represent generalized shocks.

Let $G(U_0, s, U, s_4, U_4, s_5, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot S, T_4, ..., T_n)$ near $(U_0^*, s_1^*, U_3^*, s_4^*, U_4^*, s_5^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s, U)$ is given by the left-hand side of Eq. (9.20), and $G_2(U, s_4, U_4, s_5, ..., s_n, U_n)$ is as in Step 1. The linearization of Eq. (9.20) at $(U_0^*, s_1^*, U_3^*)$ is

$$(DF(U_3^*) - s_1^* I)\dot{U} - (DF(U_0^*) - s_1^* I)\dot{U}_0 - \dot{s}(U_3^* - U_0^*) = 0. \tag{9.22}$$

Solutions of Eq. (9.22) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}, \dot{U}) = (0, 1, (DF(U_3^*) - s_1^* I)^{-1} (U_3^* - U_0^*)). \tag{9.23}$$

Therefore (R1) holds. Condition (A) then follows from assumption (3) of the theorem.

The solutions of (9.20)–(9.21) near $(U_0^*, s_1^*, U_3^*)$ are parameterized by $U_0$ and $s$ as

$$U = U(U_0, s), \qquad s \leqslant s_1(U_0), \tag{9.24}$$

where $U(U_0, s)$ is the solution of Eq. (9.20) near $(U_0^*, s_1^*, U_3^*)$.

The map $H$ for this situation is $H(U_0, s, U, s_4, ..., s_n, U_n) = s_1(U_0) - s$. Therefore

$$\tilde{H}(U_0, U_n) = s_1(U_0) - s(U_0, U_n).$$

Thus $D_{U_n}\tilde{H}(U_0^*, U_n^*) = D_{U_n} s(U_0^*, U_n^*)$, which is nonzero by the proof of Proposition 4.2. This verifies the second half of (E1).

We note that by the argument used to prove Proposition 4.3, the first half of (E1) holds if and only if there is a solution $(\dot{U}_0, \dot{s}, \dot{U})$ of Eq. (9.22) such that

(a) $\dot{U}$ is a multiple of $(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$,

(b) $\dot{s}_1 - \dot{s} = Ds_1(U_0^*)\dot{U}_0 - \dot{s} \neq 0$.

The solutions of Eq. (9.22) are triples $(\dot{U}_0, \dot{s}, \dot{U})$ with

$$\dot{U}_0 = (DF(U_0^*) - s_1^* I)^{-1} \{(DF(U_3^*) - s_1^* I)\, \dot{U} - \dot{s}(U_3^* - U_0^*)\}. \quad (9.25)$$

To calculate $\dot{s}_1 = Ds_1(U_0^*)\, \dot{U}_0$, we multiply (9.8) by $\ell_1(U_1^*)$, which yields

$$\dot{s}_1 = -\frac{\ell_1(U_1^*)(DF(U_0^*) - s_1^* I)\, \dot{U}_0}{\ell_1(U_1^*)(U_1^* - U_0^*)}. \quad (9.26)$$

Then, given a solution $(\dot{U}_0, \dot{s}, \dot{U})$ of (9.22), from (9.25) and (9.26) we calculate that

$$\dot{s}_1 - \dot{s} = -\frac{\ell_1(U_1^*)(DF(U_3^*) - s_1^* I)\, \dot{U}}{\ell_1(U_1^*)(U_1^* - U_0^*)} + \left\{ \frac{\ell_1(U_1^*)(U_3^* - U_0^*)}{\ell_1(U_1^*)(U_1^* - U_0^*)} - 1 \right\} \dot{s}. \quad (9.27)$$

Let $\dot{U} = a(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$. If there exist $(a, \dot{s})$ such that (9.27) is nonzero, then (E1) holds. If there do not exist such $(a, \dot{s})$, we derive a contradiction as follows.

If (9.27), with $\dot{U} = a(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$, is zero for all $(a, \dot{s})$, then

$$\ell_1(U_1^*)(DF(U_3^*) - s_1^* I)\, \frac{\partial \tilde{U}_3}{\partial \tau}\,(U_n^*, \tau^*) = 0 \quad (9.28)$$

and

$$\ell_1(U_1^*)(U_3^* - U_0^*) = \ell_1(U_1^*)(U_1^* - U_0^*). \quad (9.29)$$

Now in verifying (A) in Step 1, we implicitly noted that $(DF(U_3^*) - s_3^* I)^{-1}$ $(U_3^* - U_2^*)$ and $(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$ are linearly independent. Therefore $U_3^* - U_2^*$ and $(DF(U_3^*) - s_3^* I)(\partial \tilde{U}_3/\partial \tau)(U_n^*, \tau^*)$ are linearly independent, so Eq. (9.28) implies that $\ell_1(U_1^*)(U_3^* - U_2^*) \neq 0$. However, by (9.1) and (9.29),

$$\begin{aligned} \ell_1(U_1^*)(U_3^* - U_2^*) &= \ell_1(U_1^*)(U_3^* - U_1^*) \\ &= \ell_1(U_1^*)(U_3^* - U_0^* - U_1^* + U_0^*) \\ &= \ell_1(U_1^*)(U_3^* - U_0^*) - \ell_1(U_1^*)(U_1^* - U_0^*) = 0. \end{aligned}$$

This is a contradiction.

The case of $\ell_1(U_1^*)(U_1^* - U_0^*) < 0$ is similar; in (9.21) and (9.24) the inequalities are reversal.

*Step* 3. From Eq. (9.13) and Eq. (9.23) we see that (Ia) need not hold, so the join may not be regular. ∎

*Remark.* The observation that (Ia) need not hold is equivalent to the observation that the one-wave curve need not continue smoothly through the degeneracy. This is related to the fact that assumption (3) of the theorem, which does not correspond to any assumption in earlier cases, is needed in Step 2 of the proof. Assumption (3) is transversality of the $R \cdot S$ shock curve $U(U_0^*, s)$ defined by (9.24) to the backward wave curve $\tilde{U}_3(U_n^*, \tau)$.

*Remark.* If the Lax admissibility criterion, rather than the viscous profile criterion, is used, then in Fig. 9.1 the shocks from $U_0$ to the distant saddle with $s > s_1(U_0)$ become admissible. Thus the one-wave curve branches. This is usually considered a defect of the Lax criterion.

## 10. PREDECESSOR $R \cdot RS$, SUCCESSOR $RS \cdot RS$

THEOREM 10.1. *Let* (2.6) *be a Riemann problem solution of type* $(T_1, ..., T_n)$ *with* $T_1 = R \cdot RS$, $T_2 = R_1$, $T_3 = RS \cdot RS$, $T_4 = R_1$. *Assume*:

(1) *All hypotheses of Theorem* 2.16 *are satisfied, except that the rarefaction* $U_1^* \xrightarrow{s_2^*} U_2^*$ *has zero strength.*

(2) $\ell_1(U_3^*)(U_3^* - U_0^*) \neq 0$.

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(R \cdot RS, R_1, T_5, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a join that is a* $U_L$*-boundary. The join is regular* (*resp. folded*) *if*

$$\ell_1(U_1^*)(U_1^* - U_0^*) \cdot \ell_1(U_3^*)(U_3^* - U_1^*) \cdot \ell_1(U_3^*)(U_3^* - U_0^*)$$

*is positive* (*resp. negative*).

*Proof.* *Step* 1. The one-wave group of (2.6) begins

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^* \xrightarrow{s_4^*} U_4^*$$

(and may be longer). We have

$$s_1^* = s_2^* = s_3^* = \lambda_1(U_1^*) = \lambda_1(U_3^*) \qquad \text{and} \qquad U_1^* = U_2^*.$$

We note that $(U_0, s_1, ..., s_3, U_3)$ near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ represents an admissible wave sequence of type $(R \cdot RS, R_1, RS \cdot RS)$ if and only if

$$F(U_1)-F(U_0)-s_1(U_1-U_0)=0, \tag{10.1}$$

$$\lambda_1(U_1)-s_1=0, \tag{10.2}$$

$$U_2-\psi(U_1, s_2)=0, \tag{10.3}$$

$$s_2-\lambda_1(U_1)\geqslant 0, \tag{10.4}$$

$$F(U_3)-F(U_2)-s_3(U_3-U_2)=0, \tag{10.5}$$

$$\lambda_1(U_2)-s_3=0, \tag{10.6}$$

$$\lambda_1(U_3)-s_3=0. \tag{10.7}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R\cdot RS, R_1, RS\cdot RS, R_1, T_5, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G=(G_1, G_2)$, where $G_1(U_0, s_1, ..., s_3, U_3)$ is given by the left hand sides of Eqs. (10.1)–(10.3) and (10.5)–(10.7), and $G_2(U_3, s_4, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_5, ..., T_n)$. From the theory of [9],

$$DG_1(U_0^*, s_1^*, ..., s_3^*, U_3^*), \text{ restricted to}$$
$$\{(\dot U_0, \dot s_1, ..., \dot s_3, \dot U_3) : \dot U_0=0\}, \text{ is an isomorphism,} \tag{10.8}$$

and

$$DG_2(U_3^*, s_4^*, ..., s_n^*, U_n^*), \text{ restricted to}$$
$$\{(\dot U_3, \dot s_4, ..., \dot s_n, \dot U_n) : \dot U_3=\dot U_n=0\}, \text{ is an isomorphism.} \tag{10.9}$$

Therefore (A) holds.

From (10.8), we can solve Eqs. (10.1)–(10.3) and (10.5)–(10.7) for $(s_1, U_1, ..., s_3, U_3)$ in terms of $U_0$ near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$. A solution of $G=0$ represents a Riemann solution of the desired type if and only if $s_2-\lambda_1(U_1)\geqslant 0$. By the definition of $\psi$, $s_2=\lambda_1(U_2)$, and $\lambda_1(U_2)=s_3$ by Eq. (10.7); moreover, $\lambda_1(U_1)=s_1$ by Eq. (10.2). Thus we need $s_3-s_1\geqslant 0$, so we study $\tilde H(U_0):=s_3(U_0)-s_1(U_0)$. We verify (E2) by showing that $D\tilde H(U_0^*)\neq 0$. We calculate $D\tilde H(U_0^*)\dot U_0$ by linearizing Eqs. (10.1)–(10.3) and (10.5)–(10.7) at $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ and solving for $\dot s_3-\dot s_1$ in terms of $\dot U_0$.

Linearizing Eqs. (10.1)–(10.3) and (10.5)–(10.7) yields

$$(DF(U_1^*)-s_1^* I)\,\dot U_1-\dot s_1(U_1^*-U_0^*)=(DF(U_0^*)-s_1^* I)\,\dot U_0, \tag{10.10}$$

$$D\lambda_1(U_1^*)\,\dot U_1-\dot s_1=0, \tag{10.11}$$

$$\dot U_2-D\psi(U_1^*, s_1^*)(\dot U_1, \dot s_2)=0, \tag{10.12}$$

$$(DF(U_3^*) - s_1^* I)\,\dot U_3 - (DF(U_1^*) - s_1^* I)\,\dot U_2 - \dot s_3(U_3^* - U_1^*) = 0, \tag{10.13}$$

$$D\lambda_1(U_1^*)\,\dot U_2 - \dot s_3 = 0, \tag{10.14}$$

$$D\lambda_1(U_3^*)\,\dot U_3 - \dot s_3 = 0. \tag{10.15}$$

We write

$$\dot U_1 = ar_1(U_1^*) + br_2(U_1^*),$$

$$\dot U_2 = cr_1(U_1^*) + dr_2(U_1^*).$$

We multiply Eq. (10.10) by $\ell_1(U_1^*)$ and $\ell_2(U_1^*)$, Eq. (10.12) by $\ell_2(U_1^*)$, and Eq. (10.13) by $\ell_1(U_3^*)$. We get

$$-\dot s_1 \ell_1(U_1^*)(U_1^* - U_0^*) = \ell_1(U_1^*)(DF(U_0^*) - s_1^* I)\,\dot U_0,$$

$$(\lambda_2(U_1^*) - \lambda_1(U_1^*))b - \dot s_1 \ell_2(U_1^*)(U_1^* - U_0^*) = \ell_2(U_1^*)(DF(U_0^*) - s_1^* I)\,\dot U_0,$$

$$d = b,$$

$$-d\ell_1(U_3^*)(\lambda_2(U_1^*) - \lambda_1(U_1^*))\, r_2(U_1^*) - \dot s_3 \ell_1(U_1^*)(U_3^* - U_1^*) = 0.$$

From these equations we can solve for $\dot s_3 - \dot s_1$ in terms of $\dot U_0$:

$$\dot s_3 - \dot s_1 = m\dot U_0 \qquad (m \text{ a } 1 \times 2 \text{ vector}),$$

$$\begin{aligned} m = {} & \{\ell_1(U_1^*)(U_1^* - U_0^*) \cdot \ell_1(U_3^*)(U_3^* - U_1^*)\}^{-1} \\ & \times \{[\ell_1(U_3^*)\, r_2(U_1^*) \cdot \ell_2(U_1^*)(U_1^* - U_0^*) + \ell_1(U_3^*)(U_3^* - U_1^*)]\, \ell_1(U_1^*) \\ & - \ell_1(U_3^*)\, r_2(U_1^*) \cdot \ell_1(U_1^*)(U_1^* - U_0^*) \cdot \ell_2(U_1^*)\}(DF(U_0^*) - s_1^* I). \end{aligned} \tag{10.16}$$

Note that the denominator is nonzero by the nondegeneracy conditions for waves of types $R \cdot RS$ and $RS \cdot RS$.

To verify (E2) we need to show that $m \neq 0$. We note that it is enough to show that if

$$\ell_1(U_3^*)\, r_2(U_1^*) \cdot \ell_1(U_1^*)(U_1^* - U_0^*) = 0, \tag{10.17}$$

then

$$\ell_1(U_3^*)\, r_2(U_1^*) \cdot \ell_2(U_1^*)(U_1^* - U_0^*) + \ell_1(U_3^*)(U_3^* - U_1^*) \neq 0. \tag{10.18}$$

But if Eq. (10.17) holds then $\ell_1(U_3^*) r_2(U_1^*) = 0$, so Eq. (10.18) holds by the nondegeneracy conditions for waves of type $RS \cdot RS$.

Since $m \neq 0$, $\mathscr{C} = \{U_0 \colon \tilde{H}(U_0) = 0\}$ is a smooth curve near $U_0^*$, and for $(U_0, U_n)$ near $(U_0^*, U_n^*)$, a solution of type $(R \cdot RS, R_1, RS \cdot RS, R_1, T_5, ..., T_n)$ exists provided $U_0$ is on the side of $\mathscr{C}$ to which $m$ points.

*Step* 2. We consider the point $(U_0^*, s_1^*, U_3^*, s_4^*, U_4^*, s_5^*, ..., s_n^*, U_n^*)$ in $\mathbb{R}^{3n-4}$. We shall investigate the existence of nearly points $(U_0, s, U, s_4, U_4, s_5, ..., s_n, U_n)$ that represent Riemann solutions of type $(R \cdot RS, R_1, T_5, ..., T_n)$. To obtain a condition for the existence of such points, we consider the bifurcation diagram of

$$\dot{U} = F(U) - F(U_0) - s(U - U_0).$$

One possibility for this diagram for $U_0 = U_0^*$, is shown in Fig. 10.1a. Note that assumption (2) of the theorem guarantees that there is a saddle-node bifurcation at $U_3^*$. There are three other possibilities for the diagram, with the parabolic curves opening to various sides.

For $U_0$ near $U_0^*$, the bifurcation diagram of Fig. 10.1a can perturb to one of those shown in Figs. 10.1b and 10.1c. Only if the bifurcation diagram is as in Fig. 10.1b is there an $R \cdot RS$ shock from $U_0$ to $U(U_0)$. For the bifurcation diagram of Fig. 10.1a there is a generalized shock from $U_0$ to $U(U_0)$.

To study this situation we consider the systems

$$F(U_1) - F(U_0) - s_1(U_1 - U_0) = 0, \tag{10.19}$$

$$\lambda_1(U_1) - s_1 = 0, \tag{10.20}$$

and

$$F(U) - F(U_0) - s(U - U_0) = 0, \tag{10.21}$$

$$\lambda_1(U) - s = 0, \tag{10.22}$$

near $(U_0, s_1, U_1) = (U_0^*, s_1^*, U_1^*)$ and $(U_0, s, U) = (U_0^*, s_1^*, U_3^*)$ respectively. They have solutions $(s_1(U_0), U_1(U_0))$ and $(s(U_0), U(U_0))$ respectively. Let $G(U_0, s, U, s_4, U_4, s_5, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R \cdot RS, R_1, T_5, ..., T_n)$ near $(U_0^*, s_1^*, U_3^*, s_4^*, U_4^*, s_5^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s, U)$ is given by the left-hand side of Eq. (10.21), and $G_2(U, s_4, U_4, s_5, ..., s_n, U_n)$ is as in Step 1. From the proof of Lemma 5.3 in Ref. [9], $DG_1(U_0^*, s_1^*, U_3^*)$, restricted to $\{(\dot{U}_0, \dot{s}, \dot{U}) : \dot{U}_0 = 0\}$, is an isomorphism. Then from (10.9), (A) holds.

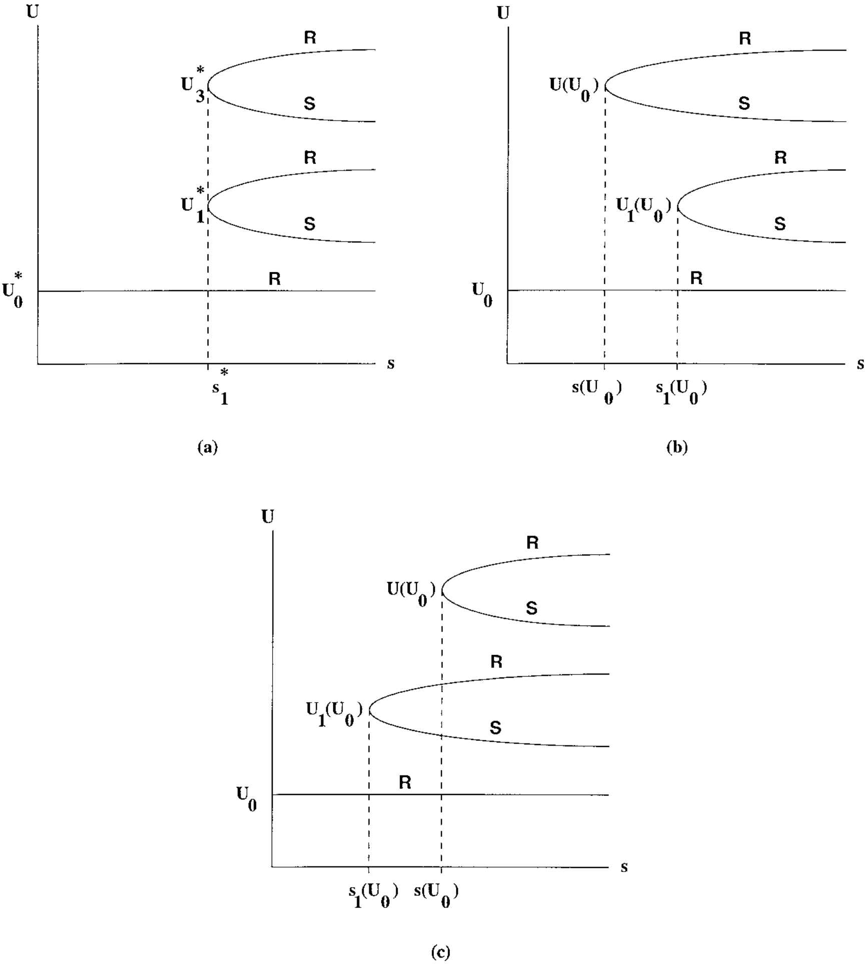


**FIG. 10.1.** Bifurcation diagrams for $\dot{U} = F(U) - F(U_0) - s(U - U_0)$, $U_0$ fixed near $U_0^*$. The proof shows that bifurcation diagram (a) actually occurs for any $U_0$ on the curve $\mathscr{C}$ defined in the proof; the other two diagrams occur for $U_0$ on opposite sides of $\mathscr{C}$.

If the lower parabola opens to the right as shown (i.e., if $\ell_1(U_1^*)(U_1^* - U_0^*) > 0$), a solution of $G = 0$ actually represents a wave sequence of the desired type if and only if $s(U_0) < s_1(U_0)$. If $\ell_1(U_1^*)(U_1^* - U_0^*) < 0$, we need $s(U_0) > s_1(U_0)$. We therefore study the function $\tilde{H}(U_0, U_n) := s_1(U_0) - s(U_0)$.

We calculate $D_{U_0}\tilde{H}(U_0^*, U_n^*)\dot{U}_0$ by linearizing Eqs. (10.19)–(10.22) at $(U_0, s_1, U_1, s, U) = (U_0^*, s_1^*, U_1^*, s_1^*, U_3^*)$ and solving for $\dot{s}_1 - \dot{s}$ in terms of $\dot{U}_0$. The result is

$$\dot{s}_1 - \dot{s} = n\dot{U}_0 \qquad (n \text{ a } 1\times 2 \text{ vector}),$$

$$\begin{aligned} n = &\{\ell_1(U_1^*)(U_1^* - U_0^*)\cdot \ell_1(U_3^*)(U_3^* - U_0^*)\}^{-1} \\ &\times \{[\ell_1(U_1^*)(U_1^* - U_0^*)\cdot \ell_1(U_3^*)\, r_1(U_1^*) - \ell_1(U_3^*)(U_3^* - U_0^*)]\, \ell_1(U_1^*) \\ &+ \ell_1(U_3^*)\, r_2(U_1^*)\cdot \ell_1(U_1^*)(U_1^* - U_0^*)\cdot \ell_2(U_1^*)\}(DF(U_0^*) - s_1^* I). \end{aligned} \tag{10.23}$$

The denominator is nonzero by the nondegeneracy conditions for waves of type $R\cdot RS$ and asumption (2) of the theorem.

We now claim

$$m = \frac{\ell_1(U_3^*)(U_3^* - U_0^*)}{\ell_1(U_3^*)(U_3^* - U_1^*)}\, n. \tag{10.24}$$

The denominator of the fraction in Eq. (10.24) is nonzero by the fact that the third wave $w_3^*$ satisfies the wave nondegeneracy conditions for waves of type $RS\cdot RS$. Thus, since $m$ is a nonzero vector, so is $n$, so (E2) holds.

To prove (10.24) we must show that

$$\begin{aligned} &\ell_1(U_3^*)\, r_2(U_1^*)\cdot \ell_2(U_1^*)(U_1^* - U_0^*) + \ell_1(U_3^*)(U_3^* - U_1^*) \\ &\quad = -\ell_1(U_1^*)(U_1^* - U_0^*)\cdot \ell_1(U_3^*)\, r_1(U_1^*) + \ell_1(U_3^*)(U_3^* - U_0^*), \end{aligned}$$

or

$$\begin{aligned} &\ell_1(U_3^*)\{\ell_2(U_1^*)(U_1^* - U_0^*)\cdot r_2(U_1^*) + \ell_1(U_1^*)(U_1^* - U_0^*)\cdot r_1(U_1^*)\} \\ &\quad = \ell_1(U_3^*)(U_1^* - U_0^*), \end{aligned}$$

which is clear.

*Step* 3. It is easy to see that $\{U_0 : \tilde{H}(U_0) = 0\}$, where $\tilde{H}$ comes from Step 2, is precisely the curve $\mathscr{C}$ defined in Step 1. For $(U_0, U_n)$ near $(U_0^*, U_n^*)$ we have the following conclusions:

(1) $\ell_1(U_1^*)(U_1^* - U_0^*) > 0$. Solutions of type $(R\cdot RS, R_1, T_5, ..., T_n)$ exist on the side of $\mathscr{C}$ to which $n$ points. Thus the join is regular (resp. folded) if $\ell_1(U_3^*)(U_3^* - U_0^*)/\ell_1(U_3^*)(U_3^* - U_1^*)$ is positive (resp. negative).

(2) $\ell_1(U_1^*)(U_1^* - U_0^*) < 0$. Solutions of type $(R\cdot RS, R_1, T_5, ..., T_n)$ exist on the side of $\mathscr{C}$ opposite that to which $n$ points. Thus the join is regular (resp. folded) if $\ell_1(U_3^*)(U_3^* - U_0^*)/\ell_1(U_3^*)(U_3^* - U_1^*)$ is negative (resp. positive). ∎

*Remark.* In contrast to Theorem 7.1, an extra assumption, assumption (2), is needed in Step 2 of the proof. This asumption is required so that the generalized shock from $U_0^*$ to $U_3^*$ ("generalized" because there is a chain of two connections joining the equilibria rather than a single connection) will nevertheless satisfy nondegeneracy condition (B1) for $R \cdot RS$ shocks.

*Remark.* If the Lax admissibility criterion is used, then in Fig. 10.1b the shocks from $U_0$ to $U(U_0)$ are also admissible.

## 11. PREDECESSOR $RS \cdot RS$, NO SUCCESSOR

THEOREM 11.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$. *Assume there is an integer k such that* $T_k = R_1$, $T_{k+1} = RS \cdot RS$, $T_{k+2} = R_1$, $T_{k+3} \neq RS \cdot *$. *Assume*:

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_{k+1}^* \xrightarrow{s_{k+2}^*} U_{k+2}^*$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_{k+2}(U_n, \tau)$ *is regular at* $(U_n^*, \tau^*)$.

(3) *The forward wave curve mapping* $U_{k+1}(U_0, s_{k+1})$ *is regular at* $(U_0^*, s_{k+1}^*)$.

(4) *In the case* $k = 1$, *the numerator of expression* (11.17) *below is nonzero.* (*If* $k > 1$, *the numerator of an analagous expression must be nonzero.*)

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(T_1, ..., T_{k-1}, R_1, RS \cdot S, T_{k+3}, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a regular join that is an intermediate boundary.*

*Proof. Step* 1. From the general theory of [9], if $k > 1$, the system of equations for wave sequences of type $(T_1, ..., T_{k-1})$ can be solved for $(s_1, U_1, ..., s_{k-1}, U_{k-1})$ in terms of $U_0$ near $(U_0^*, s_1^*, ..., s_{k-1}^*, U_{k-1}^*)$.

However, we shall assume for simplicity that $k = 1$. Thus the one-wave group of (2.6) is

$$U_0^* \xrightarrow{s_1^*} U_1^* \xrightarrow{s_2^*} U_2^* \xrightarrow{s_3^*} U_3^*,$$

with $T_1 = R_1$, $T_2 = RS \cdot RS$, $T_3 = R_1$, $T_4 \neq RS \cdot S$. We have

$$s_2^* = s_3^* = \lambda_1(U_1^*) = \lambda_1(U_2^*) \qquad \text{and} \qquad U_2^* = U_3^*.$$

We note that $(U_0, s_1, ..., s_3, U_3)$ near $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ represents an admissible wave sequence of type $(R_1, RS \cdot RS, R_1)$ if and only if

$$U_1 - \psi(U_0, s_1) = 0, \tag{11.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{11.2}$$

$$\lambda_1(U_1) - s_2 = 0, \tag{11.3}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{11.4}$$

$$U_3 - \psi(U_2, s_3) = 0, \tag{11.5}$$

$$s_3 - \lambda_1(U_2) \geqslant 0. \tag{11.6}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot RS, R_1, T_4, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, ..., s_3, U_3)$ is given by the left hand sides of Eqs. (11.1)–(11.5), and $G_2(U_3, s_4, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(T_4, ..., T_n)$. The linearization of Eqs. (11.1)–(11.5) at $(U_0^*, s_1^*, ..., s_3^*, U_3^*)$ is

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{11.7}$$

$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_2^* I)\,\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{11.8}$$

$$D\lambda_1(U_1^*)\,\dot{U}_1 - \dot{s}_2 = 0, \tag{11.9}$$

$$D\lambda_1(U_2^*)\,\dot{U}_2 - \dot{s}_2 = 0, \tag{11.10}$$

$$\dot{U}_3 - D\psi(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0. \tag{11.11}$$

Solutions of Eqs. (11.7)–(11.11) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3) = (0, 0, 0, 0, 0, 1, r_1(U_2^*)). \tag{11.12}$$

Thus (R1) holds. Since (R2) and (S3) follow from assumption (1), (A) holds.

Solutions of (11.1)–(11.6) near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$ are parameterized by $U_0$ and $s_3$ as

$$s_1 = s_1(U_0),$$

$$U_1 = U_1(U_0),$$

$$s_2 = s_2(U_0),$$

$$U_2 = U_2(U_0),$$

$$U_3 = \psi(U_2, s_3), \qquad s_3 \geqslant \lambda_1(U_2).$$

Here $(s_1, U_1, s_2, U_2) = (s_1(U_0), U_1(U_0), s_2(U_0), U_2(U_0))$ is the solution of Eqs. (11.1)–(11.4), and $s_1(U_0) = s_2(U_0) = \lambda_1(U_1(U_0)) = \lambda_1(U_2(U_0))$. We have

$$\frac{\partial U_3}{\partial s_3}(U_0, s_1(U_0)) = r_1(U_2(U_0)). \tag{11.14}$$

The left-hand side of (11.6) is the map $H$ for this situation, so $\tilde{H}(U_0, U_n) = s_3(U_0, U_n) - \lambda_1(U_2(U_0))$. The second part of (E1) holds by Proposition 4.2.

To verify the first part of (E1) using Proposition 4.3, let

$$\frac{\partial \tilde{U}_3}{\partial \tau}(U_n^*, \tau^*) = \alpha r_1(U_2^*) + \beta r_2(U_2^*). \tag{11.14}$$

We set

$$\dot{U}_1 = a r_1(U_1^*) + b r_2(U_1^*),$$

$$\dot{U}_2 = c r_1(U_2^*) + d r_2(U_2^*),$$

and, motivated by Proposition 4.3, we set

$$\dot{U}_3 = \alpha r_1(U_2^*) + \beta r_2(U_2^*).$$

We multiply Eq. (11.8) and Eq. (11.11) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then Eqs. (11.8)–(11.11) become the system

$$-\ell_1(U_2^*)(\lambda_2(U_1^*) - s_2^*)\, b r_2(U_1^*) - \dot{s}_2 \ell_1(U_2^*)(U_2^* - U_1^*) = 0,$$

$$(\lambda_2(U_2^*) - s_2^*) d - \ell_2(U_2^*)(\lambda_2(U_1^*) - s_2^*)\, b r_2(U_1^*) - \dot{s}_2 \ell_2(U_2^*)(U_2^* - U_1^*) = 0,$$

$$D\lambda_1(U_1^*)(a r_1(U_1^*) + b r_2(U_1^*)) - \dot{s}_2 = 0,$$

$$D\lambda_1(U_2^*)(c r_1(U_2^*) + d r_2(U_2^*)) - \dot{s}_2 = 0,$$

$$\dot{s}_3 - d D\lambda_1(U_2^*)\, r_2(U_2^*) = \alpha, \tag{11.15}$$

$$d = \beta. \tag{11.16}$$

We have use Lemma 2.2 in Eqs. (11.15)–(11.16).

Simplifying the notation, this system becomes

$$-Ab - B\dot{s}_2 = 0,$$

$$-Cb + Ed - G\dot{s}_2 = 0,$$

$$a + Hb - \dot{s}_2 = 0,$$

$$c + Id - \dot{s}_2 = 0,$$

$$\dot{s}_3 - Id = \alpha,$$

$$d = \beta.$$

Here $A$, $B$, $C$, $E$, $G$, $H$, and $I$ have the obvious meanings. Since $\ell_1(U_2^*)(U_2^* - U_1^*) \neq 0$ by the nondegeneracy conditions for waves of type $RS \cdot RS$, $U_2^* - U_1^*$ and $r_2(U_2^*)$ are linearly independent; this implies that $BC - GA \neq 0$. Therefore this system can be solved uniquely for $(a, b, \dot{s}_2, c, d, \dot{s}_3)$. Then we have

$$\dot{s}_3 - D\lambda_1(U_2^*)\dot{U}_2 = \dot{s}_3 - \dot{s}_2 = (\alpha + I\beta) + \frac{EA\beta}{BC - GA}$$

$$= \frac{(BC - GA)\alpha + (BCI - GAI + EA)\beta}{BC - GA}, \qquad (11.17)$$

which is nonzero by assumption (4) of the theorem. Thus the hypotheses of Proposition 4.3 are satisfied provided we can choose $(\dot{U}_0, \dot{s}_1)$ to satisfy Eq. (11.7) with $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$; since Lemma 2.2 says that $D\psi(U_0^*, s_1^*)$ is surjective, we can do this. Without the simplifying assumption $k = 1$, assumption (3) of the theorem would be required.

*Step* 2. We note that if $(U_0, s_1, U_1, s, U)$ represents an admissible wave sequence of type $(R_1, RS \cdot S)$ or $(R_1, RS \cdot RS)$, then we must have

$$U_1 - \psi(U_0, s_1) = 0, \qquad (11.18)$$

$$F(U) - F(U_1) - s(U - U_1) = 0, \qquad (11.19)$$

$$\lambda_1(U_1) - s = 0, \qquad (11.20)$$

$$\lambda_1(U) - s \leqslant 0. \qquad (11.21)$$

In fact, any solution of Eqs. (11.18)–(11.20) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ with $\lambda_1(U) - s < 0$ represents a wave sequence of type $(R_1, RS \cdot S)$, while any solution with $\lambda_1(U) - s = 0$ represents a wave sequence of type $(R_1, RS \cdot RS)$. The admissibility of the waves can be seen from the bifurcation diagram of

$$\dot{U} = (F(U) - F(U_1) - s(U - U_1)$$

for $U_1$ near $U_1^*$. See Fig. 11.1.

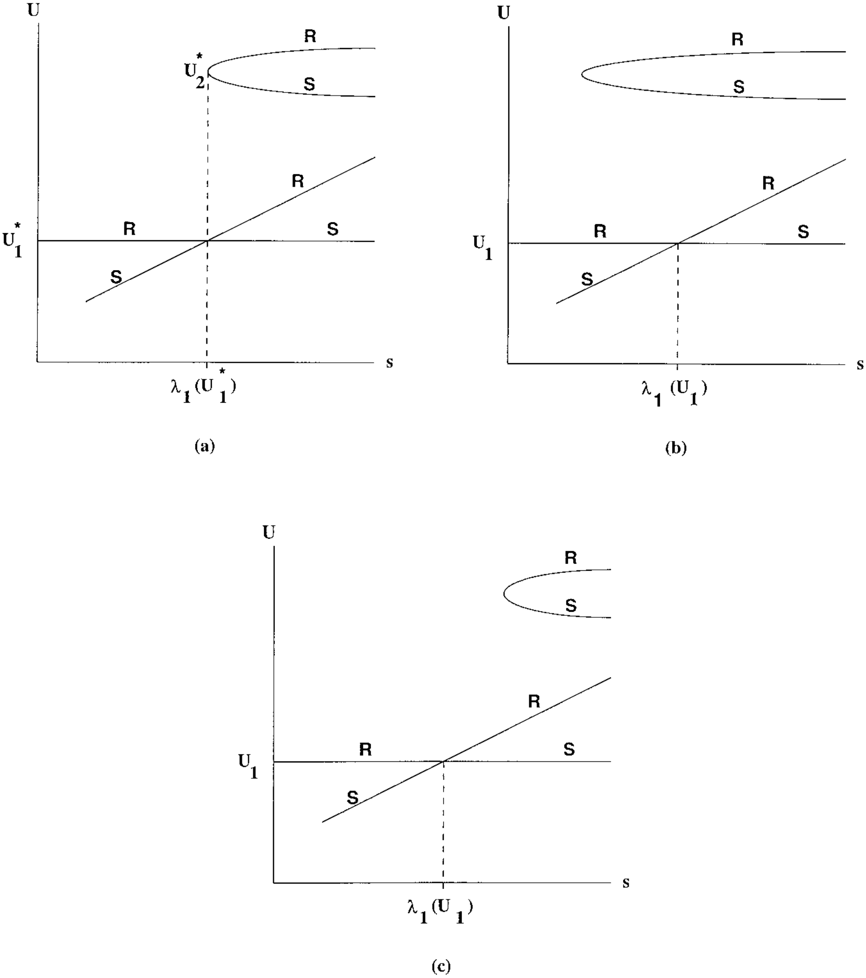


**FIG. 11.1.** Bifurcation diagrams for $\dot{U} = F(U) - F(U_1) - s(U - U_1)$, $U_1$ fixed near $U_1^*$. Only in case (b) (resp. case (a)) is there a repeller-saddle to saddle (resp. repeller-saddle to repeller-saddle) connection.

Let $G(U_0, s_1, U_1, s, U, s_4, U_4, s_5, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot S, T_4, ..., T_n)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*, s_4^*, U_4^*, s_5^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, U_1, s, U)$ is given by the left-hand sides of Eqs. (11.18)–(11.20), and $G_2(U, s_4, U_4, s_5, ..., s_n, U_n)$ is as in Step 1. The linearization of Eqs. (11.18)–(11.20) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ is

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{11.22}$$

$$(DF(U_2^*) - s_2^* I)\,\dot{U} - (DF(U_1^*) - s_2^* I)\,\dot{U}_1 - \dot{s}(U_2^* - U_1^*) = 0, \tag{11.23}$$

$$D\lambda_1(U_1^*)\,\dot{U}_1 - \dot{s} = 0. \tag{11.24}$$

Solutions of Eqs. (11.22)–(11.24) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U}) = (0, 0, 0, 0, r_1(U_2^*)). \tag{11.25}$$

Therefore (R1) holds. Since the last component of (11.12) agrees with the last component of (11.25), (A) holds.

Solutions of Eqs. (11.18)–(11.20) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$ can be parameterized by $U_0$ and a variable $t$ near 0 as

$$s_1 = s_1(U_0, t),$$

$$U_1 = U_1(U_0, t),$$

$$s = s_1(U_0, t),$$

$$U = U(U_0, t),$$

where $U_1(U_0(t)) = \psi(U_0, s_1(U_0, t))$, $s_1(U_0, t) = \lambda_1(U_1(U_0, t))$, and we can easily arrange that

$$s_1(U_0, 0) = s_1(U_0), \qquad \frac{\partial s_1}{\partial t}(U_0, 0) = 0,$$

$$U_1(U_0, 0) = U_1(U_0), \qquad \frac{\partial U_1}{\partial t}(U_0, 0) = 0,$$

$$U(U_0, 0) = U_2(U_0), \qquad \frac{\partial U}{\partial t}(U_0, 0) = r_1(U_2(U_0)).$$

From (11.21), the map $H$ for this situation is $s - \lambda_1(U)$. Therefore

$$\tilde{H}(U_0, U_n) = s(U_0, t(U_0, U_n)) - \lambda_1(U(U_0, t(U_0, U_n))).$$

The second half of (E1) is verified as in Sec. 7.

We verify the first half of (E1) using the idea of Proposition 4.3. In Eqs. (11.22)–(11.24), we set $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$ and, motivated by Proposition 4.3, we set $\dot{U} = \alpha r_1(U_2^*) + \beta r_2(U_2^*)$, with $\alpha$ and $\beta$ given by Eq. (11.14).

We multiply Eq. (11.23) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then Eqs. (11.23)–(11.24) become the system

$$-Ab - B\dot{s} = 0,$$
$$-Cb + E\beta - G\dot{s} = 0,$$
$$a + Hb - \dot{s} = 0.$$

Here the capital letters have the same meaning as in Step 1 of the proof.

This system can be solved uniquely for $(a, b, \dot{s})$. We find

$$\dot{s} - D\lambda_1(U_2^*)\dot{U} = \frac{EA\beta}{GA - BC} - (\alpha + I\beta)$$
$$= \frac{(BC - GA)\alpha + (BCI - GAI + EA)\beta}{GA - BC},$$

which is nonzero by assumption (4).

*Step* 3. In Step 1,

$$U_3 = \psi(U_2(U_0), s_3)$$

is defined for $s_3 \geqslant \lambda_1(U_2(U_0))$, while in Step 2, $U(U_0, t)$ is defined for $t \leqslant 0$ as in Section 7. Then from (11.12) and (11.25), (Ia) holds, so the join is regular. ∎

*Remark.* The triples $(U_1, s_2, U_2)$ such that there is a shock of type $RS \cdot RS$ from $U_1$ to $U_2$ with speed $s_2$ form a curve $\mathscr{D}$ through $(U_1^*, s_2^*, U_2^*)$: the solutions of Eqs. (11.2)–(11.4). This curve projects to curves $\mathscr{D}_1$ and $\mathscr{D}_2$ through $U_1^*$ and $U_2^*$ respectively: for $U_1 \in \mathscr{D}_1$, there is a speed $s_2$ and a point $U_2 \in \mathscr{D}_2$ such that there is an $RS \cdot RS$ shock from $U_1$ to $U_2$ with speed $s_2$. Assumption (4) says that $\mathscr{D}_2$ is transverse to the backward wave curve $\tilde{U}_3(U_n^*, \tau)$ at $U_2^* = U_3^*$. This is a natural geometric requirement for the existence of a codimension-one Riemann solution of the desired type. Nevertheless, it is used only to verify the first part of (E1), which does not seem to be very important to students of Riemann problems. It is interesting to note that the same assumption verifies the first part of (E1) in both steps of the proof.

## 12. PREDECESSOR $RS \cdot RS$, SUCCESSOR $RS \cdot S$

THEOREM 12.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$. *Assume there is an integer k such that* $T_k = R_1$, $T_{k+1} = RS \cdot RS$, $T_{k+2} = R_1$, $T_{k+3} = RS \cdot S$. *Assume*

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U^*_{k+1} \xrightarrow{s^*_{k+2}} U^*_{k+2}$ *has zero strength.*

(2) *The backward wave curve mapping* $\tilde{U}_{k+3}(U_n, \tau)$ *is regular at* $(U^*_n, \tau^*)$.

(3) *The forward wave curve mapping* $U_{k+1}(U_0, s_{k+1})$ *is regular at* $(U^*_0, s^*_{k+1})$.

(4) $(DF(U^*_{k+3}) - s^*_{k+1} I)^{-1}(U^*_{k+3} - U^*_k)$ *and* $(\partial \tilde{U}_{k+3}/\partial\tau)(U^*_n, \tau^*)$ *are linearly independent.*

(5) *In the case* $k=1$, *the numerator of expression* (12.20) *below is nonzero.* (*If* $k>1$, *the numerator of an analagous expression must be nonzero.*)

(6) $\ell_1(U^*_{k+2})(U^*_{k+3} - U^*_{k+2})$ *is nonzero.*

(7) $\ell_1(U^*_k)(U^*_{k+3} - U^*_k)$ *is nonzero.*

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(T_1, ..., T_{k-1}, R_1, RS \cdot S, T_{k+4}, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a join that is an intermediate boundary. The join may be regular or folded.*

*Proof. Step* 1. As in Section 11, we shall assume for simplicity that $k=1$. Then the 1-wave group of (2.6) is

$$U^*_0 \xrightarrow{s^*_1} U^*_1 \xrightarrow{s^*_2} U^*_2 \xrightarrow{s^*_3} U^*_3 \xrightarrow{s^*_4} U^*_4$$

with $T_1 = R_1$, $T_2 = RS \cdot RS$, $T_3 = R_1$, $T_4 = RS \cdot S$. We have

$$s^*_2 = s^*_3 = s^*_4 = \lambda_1(U^*_1) = \lambda_1(U^*_2) \qquad \text{and} \qquad U^*_2 = U^*_3.$$

We note that $(U_0, s_1, ..., s_4, U_4)$ near $(U^*_0, s^*_1, ..., s^*_4, U^*_4)$ represents an admissible wave sequence of type $(R_1, RS \cdot RS, R_1, RS \cdot S)$ if and only if

$$U_1 - \psi(U_0, s_1) = 0, \tag{12.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{12.2}$$

$$\lambda_1(U_1) - s_2 = 0, \tag{12.3}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{12.4}$$

$$U_3 - \psi(U_2, s_3) = 0, \tag{12.5}$$

$$s_3 - \lambda_1(U_2) \geqslant 0, \tag{12.6}$$

$$F(U_4) - F(U_3) - s_4(U_4 - U_3) = 0, \tag{12.7}$$

$$\lambda_1(U_3) - s_4 = 0. \tag{12.8}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot RS, R_1, RS \cdot S, T_5, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, ..., s_4, U_4)$ is given by the left hand sides of Eqs. (12.1)–(12.5) and (12.7)–(12.8), and $G_2(U_4, s_5, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(T_5, ..., T_n)$. The linearization of Eqs. (12.1)–(12.5) and (12.7)–(12.8) at $(U_0^*, s_1^*, ..., s_4^*, U_4^*)$ is

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \qquad (12.9)$$

$$(DF(U_2^*) - s_2^* I)\,\dot{U}_2 - (DF(U_1^*) - s_2^* I)\,\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \qquad (12.10)$$

$$D\lambda_1(U_1^*)\,\dot{U}_1 - \dot{s}_2 = 0, \qquad (12.11)$$

$$D\lambda_1(U_2^*)\,\dot{U}_2 - \dot{s}_2 = 0, \qquad (12.12)$$

$$\dot{U}_3 - D\psi(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0, \qquad (12.13)$$

$$(DF(U_4^*) - s_4^* I)\,\dot{U}_4 - (DF(U_3^*) - s_4^* I)\,\dot{U}_3 - \dot{s}_4(U_4^* - U_3^*) = 0, \qquad (12.14)$$

$$D\lambda_1(U_3^*)\dot{U}_3 - \dot{s}_4 = 0. \qquad (12.15)$$

Solutions of Eqs. (12.9)–(12.15) with $\dot{U}_0 = 0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}_2, \dot{U}_2, \dot{s}_3, \dot{U}_3, \dot{s}_4, \dot{U}_4)$$
$$= (0, 0, 0, 0, 0, 1, r_1(U_2^*), 1, (DF(U_4^*) - s_4^* I)^{-1}(U_4^* - U_3^*)). \qquad (12.16)$$

Thus (R1) holds. Since (R2) and (S3) follow from assumption (1), (A) holds.

Solutions of (12.1)–(12.6) near $(U_0^*, s_1^*, ..., s_4^*, U_4^*)$ are parameterized by $U_0$ and $s_3$ as

$$s_1 = s_1(U_0),$$

$$U_1 = U_1(U_0),$$

$$s_2 = s_2(U_0),$$

$$U_2 = U_2(U_0),$$

$$U_3 = \psi(U_2(U_0), s_3), \qquad s_3 \geqslant \lambda_1(U_2(U_0)),$$

$$s_4 = s_3,$$

$$U_4 = \varphi(U_3, s_4).$$

Here $(s_1(U_0), U_1(U_0), s_2(U_0), U_2(U_0))$ is the solution of Eqs. (12.1)–(12.4) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*)$, and $U_4 = \varphi(U_3, s_4)$ is the solution of Eq. (12.7) near $(U_3^*, s_4^*, U_4^*)$.

The left-hand side of (12.6) is the map $H$ for this situation, so $\tilde{H}(U_0, U_n) = s_3(U_0, U_n) - \lambda_1(U_2(U_0))$. The second part of (E1) holds by assumption (2) and Proposition 4.2.

To verify the first part of (E1) using Proposition 4.3, let

$$\frac{\partial \tilde{U}_4}{\partial \tau}(U_n^*, \tau^*) = \alpha r_1(U_4^*) + \beta r_2(U_4^*).$$

We set

$$\dot{U}_1 = a r_1(U_1^*) + b r_2(U_1^*),$$

$$\dot{U}_2 = c r_1(U_2^*) + d r_2(U_2^*),$$

$$\dot{U}_3 = e r_1(U_3^*) + f r_2(U_3^*),$$

and, motivated by Proposition 4.3, we set

$$\dot{U}_4 = \alpha r_1(U_4^*) + \beta r_2(U_4^*).$$

We multiply Eq. (12.10) and Eqs. (12.13)–(12.14) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then Eqs. (12.10)–(12.15) become the system

$$-\ell_1(U_2^*)(\lambda_2(U_1^*) - s_2^*)\, b r_2(U_1^*) - \dot{s}_2 \ell_1(U_2^*)(U_2^* - U_1^*) = 0,$$

$$(\lambda_2(U_2^*) - s_2^*) d - \ell_2(U_2^*)(\lambda_2(U_1^*) - s_2^*)\, b r_2(U_1^*) - \dot{s}_2 \ell_2(U_2^*)(U_2^* - U_1^*) = 0,$$

$$D\lambda_1(U_1^*)(a r_1(U_1^*) + b r_2(U_1^*)) - \dot{s}_2 = 0,$$

$$D\lambda_1(U_2^*)(c r_1(U_2^*) + d r_2(U_2^*)) - \dot{s}_2 = 0,$$

$$\dot{s}_3 - d D\lambda_1(U_2^*)\, r_2(U_2^*) - e = 0, \tag{12.17}$$

$$d - f = 0, \tag{12.18}$$

$$\ell_1(U_3^*)(\lambda_1(U_4^*) - s_4^*)\, \alpha r_1(U_4^*) + \ell_1(U_3^*)(\lambda_2(U_4^*) - s_4^*)\, \beta r_2(U_4^*) - \dot{s}_4 \ell_1(U_3^*)(U_4^* - U_3^*) = 0,$$

$$\ell_2(U_3^*)(\lambda_1(U_4^*) - s_4^*)\, \alpha r_1(U_4^*) + \ell_2(U_3^*)(\lambda_2(U_4^*) - s_4^*)\, \beta r_2(U_4^*) - (\lambda_2(U_3^*) - s_4^*) f - \dot{s}_4 \ell_2(U_3^*)(U_4^* - U_3^*) = 0,$$

$$e + f\, D\lambda_1(U_3^*)\, r_2(U_3^*) - \dot{s}_4 = 0.$$

We have use Lemma 2.2 in Eqs. (12.17)–(12.18).

Simplifying the notation, this system becomes

$$\begin{aligned} -Ab-B\dot{s}_2&=0, \\ -Cb+Ed-G\dot{s}_2&=0, \\ a+Hb-\dot{s}_2&=0, \\ c+Id-\dot{s}_2&=0, \\ \dot{s}_3-Id-e&=0, \\ d-f&=0, \\ -L\dot{s}_4&=-J\alpha-K\beta, \\ -Ef-Q\dot{s}_4&=-M\alpha-N\beta, \\ e+If-\dot{s}_4&=0. \end{aligned} \tag{12.19}$$

Here the capital letters have the obvious meanings. As in Section 11, $BC-GA\neq 0$, and by assumption (6) of the theorem $L\neq 0$. Therefore this system can be solved uniquely for $(a, b, \dot{s}_2, c, d, \dot{s}_3, e, f, \dot{s}_4)$. Then we have

$$\begin{aligned} \dot{s}_3-D\lambda_1(U_2^*)\,\dot{U}_2&=\dot{s}_3-\dot{s}_2 \\ &=\frac{\begin{bmatrix}(BCJ-AGJ-AQJ+ALM)\alpha \\ \quad +(BCK-AGK-AKQ+ALN)\beta\end{bmatrix}}{L(CB-GA)}. \end{aligned} \tag{12.20}$$

This is nonzero by assumption (5). Thus, as in Section 11, the hypotheses of Proposition 4.3 are satisfied.

*Step* 2. We look for points $(U_0, s_1, U_1, s, U)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_4^*)$ that represent admissible wave sequences of type $(R_1, RS\cdot S)$. We first note that Eqs. (12.2) and (12.4) can be solved for $(s_2, U_2)$ in terms of $U_1$ near $(U_1^*, s_2^*, U_2^*)$ by the implicit function theorem:

$$\begin{aligned} s_2&=\hat{s}_2(U_1)=\lambda_1(U_2(U_1)), \\ U_2&=\hat{U}_2(U_1). \end{aligned}$$

Next we note that for fixed $U_1$ near $U_1^*$, the one-parameter family of differential equations

$$\dot{U}=F(U)-F(U_1)-s(U-U_1) \tag{12.21}$$

has a transcritical bifurcation at $s=\lambda_1(U_1)$ and a saddle-node bifurcation at $s=\hat{s}_2(U_1)$.

The possible bifurcation diagrams for (12.21) with the parabolic curve through $U_2(U_1)$ opening to the right are shown in Fig. 12.1; the upper curve represents equilibria near $U_4^*$. The parabolic curve opens to the right (resp. left) if $\ell_1(U_2^*)(U_2^*-U_1^*)$ is positive (resp. negative).

In the case $\ell_1(U_1^*)(U_1^*-U_0^*)>0$, if $(U_0, s_1, U_1, s, U)$ represents an admissible wave sequence of type $(R_1, RS\cdot S)$, then

$$U_1-\psi(U_0, s_1)=0, \tag{12.22}$$

$$F(U)-F(U_1)-s(U-U_1)=0, \tag{12.23}$$

$$\lambda_1(U_1)-s=0. \tag{12.24}$$

In addition, from Fig. 12.1, we must have

$$\hat{s}_2(U_1)-\lambda_1(U_1)\geqslant 0. \tag{12.25}$$

In fact, solutions of Eqs. (12.22)–(12.24) near $(U_0^*, s_1^*, U_1^*, s_2^*, U_4^*)$ with $\hat{s}_2(U_1)-\lambda_1(U_1)>0$ represent wave sequences in which the second wave is an $RS\cdot S$ shock; if $\hat{s}_2(U_1)-\lambda_1(U_1)=0$, the second wave is a generalized shock.

Let $G(U_0, s_1, U_1, s, U, s_5, U_5, s_6, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS\cdot S, T_5, ..., T_n)$ near $(U_0^*, s_1^*, U_1^*, s_2^*, U_4^*, s_5^*, U_5^*, s_6^*, ..., s_n^*, U_n^*)$, $G=(G_1, G_2)$, where $G_1(U_0, s_1, U_1, s, U)$ is given by the left-hand sides of Eqs. (12.22)–(12.24), and $G_2(U, s_5, U_5, s_6, ..., s_n, U_n)$ is as in Step 1. The linearization of Eqs. (12.22)–(12.24) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_4^*)$ is

$$\dot{U}_1-D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1)=0, \tag{12.26}$$

$$(DF(U_4^*)-s_2^*I)\,\dot{U}-(DF(U_1^*)-s_2^*I)\,\dot{U}_1-\dot{s}(U_4^*-U_1^*)=0, \tag{12.27}$$

$$D\lambda_1(U_1^*)\,\dot{U}_1-\dot{s}=0. \tag{12.28}$$

Solutions of Eqs. (12.26)–(12.28) with $\dot{U}_0=0$ form a one-dimensional space spanned by

$$(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})=(0, 1, r_1(U_1^*), 1, (DF(U_4^*)-s_2^*I)^{-1}(U_4^*-U_1^*)). \tag{12.29}$$

Therefore (R1) holds. By assumption (4) of the theorem, (A) holds.

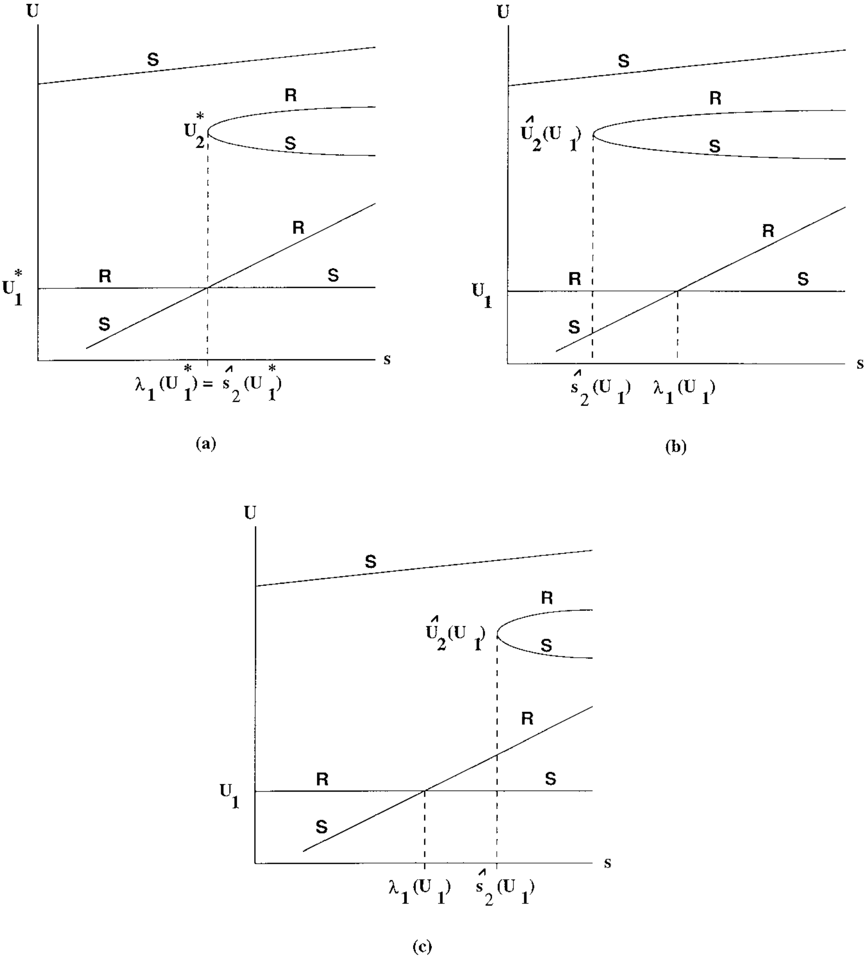


**FIG. 12.1.** Bifurcation diagrams for $\dot{U} = F(U) - F(U_1) - s(U - U_1)$, $U_1$ fixed near $U_1^*$. Only in case (c) is there a repeller-saddle to saddle connection from $U_1$ to an equilibrium near $U_4^*$.

In the case $\ell_1(U_2^*)(U_2^* - U_1^*) > 0$ solutions of Eqs. (12.22)–(12.24) near $U_0^*, s_1^*, U_1^*, s_2^*, U_4^*$), are parameterized by $U_0$ and $s_1$ as

$$U_1 = \psi(U_0, s_1),$$

$$s = s_1,$$

$$U = \eta(U_1, s),$$

where $\eta(U_1, s)$ is the solution of Eq. (12.23) near $(U_1^*, s_2^*, U_4^*)$.

The map $H$ for this situation is the left-hand side of (12.25), so

$$\tilde{H}(U_0, U_n) = \hat{s}_2(U_1) - \lambda_1(U_1), \qquad \text{where} \quad U_1 = \psi(U_0, s_1(U_0, U_n)). \tag{12.30}$$

To verify the second half of (E1), we note that

$$D_{U_n}\tilde{H}(U_0^*, U_n^*)\,\dot{U}_n = D\hat{s}_2(U_1^*)\,\dot{U}_1 - D\lambda_1(U_1^*)\,\dot{U}_1,$$

$$\dot{U}_1 = \frac{\partial\psi}{\partial s_1}(U_0^*, s_1^*)\,D_{U_n}s_1(U_0^*, U_n^*)\,\dot{U}_n.$$

By the proof of Proposition 4.2 we can choose $\dot{U}_n$ so that $D_{U_n}s_1(U_0^*, U_n^*)\,\dot{U}_n = 1$; then $\dot{U}_1 = (\partial U/\partial s_1)(U_0^*, s_1^*) = r_1(U_1^*)$. To calculate $D\hat{s}_2(U_1^*)\,r_1(U_1^*)$, we linearize Eqs. (12.2) and (12.4) at $(U_1^*, s_2^*, U_2^*)$

$$(DF(U_2^*) - s_2^* I)\,\dot{U}_2 - (DF(U_1^*) - s_2^* I)\,\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{12.31}$$

$$D\lambda_1(U_2^*)\,\dot{U}_2 - \dot{s}_2 = 0.$$

If we set $\dot{U}_1 = r_1(U_1^*)$ and multiply Eq. (12.31) by $\ell_1(U_2^*)$, we obtain $-\dot{s}_2\ell_1(U_2^*)(U_2^* - U_1^*) = 0$, so $\dot{s}_2 = 0$. Therefore

$$D\hat{s}_2(U_1^*)\,r_1(U_1^*) = 0.$$

We conclude that for $\dot{U}_n$ chosen as above,

$$D_{U_n}\tilde{H}(U_0^*, U_n^*)\,\dot{U}_n = -D\lambda_1(U_1^*)\,r_1(U_1^*) = -1.$$

Therefore the second half of (E1) holds.

To verify the first half of (E1), we note that by the argument used to prove Proposition 4.3, the first half of (E1) holds if and only if there is a solution $(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ of Eqs. (12.26)–(12.28) such that

(a) $\dot{U}$ is a multiple of $(\partial\tilde{U}_4/\partial\tau)(U_n^*, \tau^*)$,

(b) $\dot{s}_2 - D\lambda_1(U_1^*)\,\dot{U}_1 = D\hat{s}_2(U_1^*)\,\dot{U}_1 - D\lambda_1(U_1^*)\,\dot{U}_1 \neq 0$.

In the system Eqs. (12.27)–(12.28), we set $\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*)$ and, motivated by Proposition 4.3, we set $\dot{U} = \alpha r_1(U_4^*) + \beta r_2(U_4^*)$.

To find the solutions $(\dot{U}_0, \dot{s}_1, \dot{U}_1, \dot{s}, \dot{U})$ of Eqs. (12.26)–(12.28) that satisfy (a), we multiply Eq. (12.27) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$. Then Eqs. (12.27)–(12.28) become the system

$$-\ell_1(U_2^*)(\lambda_1(U_4^*)-s_2^*)\,\alpha r_1(U_4^*)+\ell_1\mathrm{AAAF}U_2^*)(\lambda_2(U_4^*)-s_2^*)\,\beta r_2(U_4^*)$$
$$-\ell_1(U_2^*)(\lambda_2(U_1^*)-s_2^*)\,br_2(U_1^*)-\dot{s}\,\ell_1(U_2^*)(U_4^*-U_1^*)=0, \tag{12.32}$$
$$-\ell_2(U_2^*)(\lambda_1(U_4^*)-s_2^*)\,\alpha r_1(U_4^*)+\ell_2(U_2^*)(\lambda_2(U_4^*)-s_2^*)\,\beta r_2(U_4^*)$$
$$-\ell_2(U_2^*)(\lambda_2(U_1^*)-s_2^*)\,br_2(U_1^*)-\dot{s}\ell_2(U_2^*)(U_4^*-U_1^*)=0, \tag{12.33}$$
$$D\lambda_1(U_1^*)(ar_1(U_1^*)+br_2(U_1^*))-\dot{s}=0. \tag{12.34}$$

Since

$$\ell_1(U_2^*)(U_4^*-U_1^*)=\ell_1(U_2^*)(U_4^*-U_3^*)+\ell_1(U_2^*)(U_3^*-U_1^*)=L+B$$

and

$$\ell_2(U_2^*)(U_4^*-U_1^*)=\ell_2(U_2^*)(U_4^*-U_3^*)+\ell_2(U_2^*)(U_3^*-U_1^*)=Q+G,$$

Eqs. (12.32)–(12.34) become the system

$$\begin{aligned}
&J\alpha+K\beta-Ab-(L+B)\dot{s}=0,\\
&N\alpha+N\beta-Cb-(Q+G)\dot{s}=0,\\
&a+Hb-\dot{s}=0.
\end{aligned}$$

Assumption (7) of the theorem implies that $r_2(U_1^*)$ and $U_4^*-U_1^*$ are linearly independent, which implies that $A(Q+G)-C(L+B)$ is nonzero. Therefore this system can be solved uniquely for $(a, b, \dot{s})$, and then $(\dot{U}_0, \dot{s}_1)$ can be easily found.

In terms of this solution, we calculate from Eq. (12.19)

$$D\hat{s}_2(U_1^*)\,\dot{U}_1=-\frac{A}{B}\,b. \tag{12.35}$$

Then we verify (b) as

$$\begin{aligned}
&D\hat{s}_2(U_1^*)\,\dot{U}_1-D\lambda_1(U_1^*)\,\dot{U}_1\\
&\quad=-\frac{A}{B}\,b-(a+Hb)\\
&\quad=\frac{\begin{bmatrix}(BCJ-AGJ-AQJ+ALM)\alpha\\ \qquad+(BCK-AGK-AKQ+ALN)\beta\end{bmatrix}}{B(A(Q+G)-C(L+B))}.
\end{aligned} \tag{12.36}$$

This expression is nonzero by assumption (5).

*Step* 3. From (12.16) and (12.29) we see that (Ia) need not hold, so the join may not be regular.

The case $\ell_1(U_1^*)(U_1^* - U_0^*) < 0$ is similar. ∎

*Remark.* As in Section 9, the observation that (Ia) need not hold is equivalent to the observation that the one-wave curve need not continue smoothly through the degeneracy, which is related to the need for assumption (4) of the theorem in Step 2 of the proof. This assumption is transversality of the $RS \cdot S$ shock curve $U = \eta(U_1^*, s)$, defined in Step 2, to the backward wave curve $\tilde{U}_4(U_n^*, \tau)$.

*Remark.* Assumption (5) has the following geometric interpretation. The quadruples $(U_1, s_2, U_2, U_4)$ such that there is a shock of type $RS \cdot RS$ from $U_1$ to $U_2$ with speed $s_2$, and a shock of type $RS \cdot S$ from $U_2$ to $U_4$ with the same speed, form a curve $\mathscr{D}$ through $(U_1^*, s_2^*, U_2^*, U_4^*)$. This curve may be found by solving Eqs. (12.2)–(12.4) and Eq. (12.7) with $U_3 = U_2$ and $s_4 = s_2$. This curve projects to curves $\mathscr{D}_1$, $\mathscr{D}_2$, and $\mathscr{D}_4$ through $U_1^*$, $U_2^*$, and $U_4^*$ respectively: for each $U_1 \in \mathscr{D}_1$ there is a speed $s_2$ and points $U_2 \in \mathscr{D}_2$ and $U_4 \in \mathscr{D}_4$ such that there is an $RS \cdot RS$ shock from $U_1$ to $U_2$ with speed $s_2$ and an $RS \cdot S$ shock from $U_2$ to $U_4$ with the same speed. Assumption (5) says that $\mathscr{D}_4$ is transverse to the backward wave curve $\tilde{U}_4(U_n^*, \tau)$ at $U_4^*$. As in the previous section, this is a natural geometric requirement for the existence of a codimension-one Riemann solution of the desired type, but it is only used to verify the first part of (E1) in both steps of the proof. Assumptions (6) and (7) are used in the verification of (E1) in Steps 1 and 2, respectively, and are analogues. However, we do not have a natural geometric interpretation for these assumptions.

*Remark.* If the Lax admissibility criterion is used, then in Fig. 12.1b the $RS \cdot S$ shocks from $U_1$ to the distant saddle become admissible.

## 13. PREDECESSOR $RS \cdot RS$, SUCCESSOR $RS \cdot RS$

THEOREM 13.1. *Let* (2.6) *be a Riemann solution of type* $(T_1, ..., T_n)$. *Assume there is an integer* $k$ *such that* $T_k = R_1$, $T_{k+1} = RS \cdot RS$, $T_{k+2} = R_1$, $T_{k+3} = RS \cdot RS$, $T_{k+4} = R_1$. *Assume*:

(1) *All hypotheses of Theorem* 2.4 *are satisfied, except that the rarefaction* $U_{k+1}^* \xrightarrow{s_{k+2}^*} U_{k+2}^*$ *has zero stength.*

(2) $\ell_1(U_{k+3}^*)(U_{k+3}^* - U_k^*) \neq 0$.

(3) $-\ell_1(U^*_{k+3})\, r_2(U^*_k) \cdot \ell_1(U^*_{k+1})(U^*_{k+1} - U^*_k) + \ell_1(U^*_{k+1})\, r_2(U^*_k) \cdot \ell_1(U^*_{k+3})(U^*_{k+3} - U^*_k) \neq 0$.

(4) *The forward wave curve mapping* $U_{k+1}(U_0, s_{k+1})$ *is regular at* $(U^*_0, s^*_{k+1})$.

*Then* (2.6) *is a codimension-one Riemann solution. It has an equivalent codimension-one Riemann solution that lies in the boundary of structurally stable Riemann solutions of type* $(T_1, ..., T_{k-1}, R_1, RS \cdot RS, R_1, T_{k+5}, ..., T_n)$. *Riemann solution* (2.6) (*and its equivalent*) *lies in a join that is a* $U_L$*-boundary. The join is regular* (*resp. folded*) *if*

$$\ell_1(U^*_{k+1})(U^*_{k+1} - U^*_k) \cdot \ell_1(U^*_{k+3})(U^*_{k+3} - U^*_{k+2}) \cdot \ell_1(U^*_{k+3})(U^*_{k+3} - U^*_k)$$

*is positive* (*resp. negative*).

*Proof. Step* 1. As in Sections 11–12, we shall assume for simplicity that $k = 1$. Then the 1-wave group of (2.6) begins

$$U^*_0 \xrightarrow{s^*_1} U^*_1 \xrightarrow{s^*_2} U^*_2 \xrightarrow{s^*_3} U^*_3 \xrightarrow{s^*_4} U^*_4 \xrightarrow{s^*_5} U^*_5$$

with $T_1 = R_1$, $T_2 = RS \cdot RS$, $T_3 = R_1$, $T_4 = RS \cdot RS$, $T_5 = R_1$; it may be longer. We have

$$s^*_2 = s^*_3 = s^*_4 = \lambda_1(U^*_1) = \lambda_1(U^*_2) = \lambda_1(U^*_4) \qquad \text{and} \qquad U^*_2 = U^*_3. \tag{13.3}$$

We note that $(U_0, s_1, ..., s_4, U_4)$ near $(U^*_0, s^*_1, ..., s^*_4, U^*_4)$ represents an admissible wave sequence of type $(R_1, RS \cdot RS, R_1, RS \cdot RS)$ if and only if

$$U_1 - \psi(U_0, s_1) = 0, \tag{13.1}$$

$$F(U_2) - F(U_1) - s_2(U_2 - U_1) = 0, \tag{13.2}$$

$$\lambda_1(U_1) - s_2 = 0, \tag{13.3}$$

$$\lambda_1(U_2) - s_2 = 0, \tag{13.4}$$

$$U_3 - \psi(U_2, s_3) = 0, \tag{13.5}$$

$$s_3 - \lambda_1(U_2) \geqslant 0 \tag{13.6}$$

$$F(U_4) - F(U_3) - s_4(U_4 - U_3) = 0, \tag{13.7}$$

$$\lambda_1(U_3) - s_4 = 0, \tag{13.8}$$

$$\lambda_1(U_4) - s_4 = 0. \tag{13.9}$$

Let $G(U_0, s_1, ..., s_n, U_n)$ be the local defining map for wave sequences of type $(R_1, RS \cdot RS, R_1, RS \cdot RS, R_1, T_6, ..., T_n)$ near $(U_0^*, s_1^*, ..., s_n^*, U_n^*)$, $G = (G_1, G_2)$, where $G_1(U_0, s_1, ..., s_4, U_4)$ is given by the left hand sides of Eqs. (13.1)–(13.5) and (13.7)–(13.9), and $G_2(U_4, s_5, ..., s_n, U_n)$ is the local defining map for wave sequences of type $(R_1, T_6, ..., T_n)$. From the theory of [9],

$$\begin{aligned}&DG_1(U_0^*, s_1^*, ..., s_4^*, U_4^*), \text{ restricted to}\\ &\qquad \{(\dot{U}_0, \dot{s}_1, ..., \dot{s}_4, \dot{U}_4) : \dot{U}_0 = 0\}, \text{ is an isomorphism,}\end{aligned} \tag{13.10}$$

and

$$\begin{aligned}&DG_2(U_4^*, s_5^*, ..., s_n^*, U_n^*), \text{ restricted to}\\ &\qquad \{(\dot{U}_4, \dot{s}_5, ..., \dot{s}_n, \dot{U}_n) : \dot{U}_4 = \dot{U}_n = 0\}, \text{ is an isomorphism.}\end{aligned} \tag{13.11}$$

Therefore (A) holds.

From (13.10), we can solve Eqs. (13.1)–(13.5) and (13.7)–(13.9) for $(s_1, U_1, ..., s_4, U_4)$ in terms of $U_0$ near $(U_0^*, s_1^*, ..., s_4^*, U_4^*)$. Since a solution of $G = 0$ represents a Riemann solution of the desired type if and only if $s_3 - \lambda_1(U_2) = s_4 - s_2 \geqslant 0$, we now study $\tilde{H}(U_0) := s_4(U_0) - s_2(U_0)$. To verify (E2), we calculate $D_{U_0}\tilde{H}(U_0^*, U_n^*)\dot{U}_0$ by linearizing Eqs. (13.1)–(13.5) and (13.7)–(13.9) at $(U_0^*, s_1^*, ..., s_4^*, U_4^*)$ and solving for $\dot{s}_4 - \dot{s}_2$ in terms of $\dot{U}_0$.

Linearizing Eqs. (13.1)–(13.5) and (13.7)–(13.9) yields:

$$\dot{U}_1 - D\psi(U_0^*, s_1^*)(\dot{U}_0, \dot{s}_1) = 0, \tag{13.12}$$

$$(DF(U_2^*) - s_2^* I)\dot{U}_2 - (DF(U_1^*) - s_1^* I)\dot{U}_1 - \dot{s}_2(U_2^* - U_1^*) = 0, \tag{13.13}$$

$$D\lambda_1(U_1^*)\dot{U}_1 - \dot{s}_2 = 0,$$

$$D\lambda_1(U_2^*)\dot{U}_2 - \dot{s}_2 = 0, \tag{13.14}$$

$$\dot{U}_3 - D\psi(U_2^*, s_3^*)(\dot{U}_2, \dot{s}_3) = 0, \tag{13.15}$$

$$(DF(U_4^*) - s_4^* I)\dot{U}_4 - (DF(U_3^*) - s_4^* I)\dot{U}_3 - \dot{s}_4(U_4^* - U_3^*) = 0, \tag{13.16}$$

$$D\lambda_1(U_3^*)\dot{U}_3 - \dot{s}_4 = 0,$$

$$D\lambda_1(U_4^*)\dot{U}_4 - \dot{s}_4 = 0.$$

We write

$$\dot{U}_1 = ar_1(U_1^*) + br_2(U_1^*), \tag{13.17}$$

$$\dot{U}_2 = cr_1(U_2^*) + dr_2(U_2^*), \tag{13.18}$$

$$\dot{U}_3 = er_1(U_3^*) + fr_2(U_3^*). \tag{13.19}$$

We multiply Eq. (13.13) by $\ell_1(U_2^*)$ and $\ell_2(U_2^*)$, Eq. (13.15) by $\ell_2(U_3^*)$, and Eq. (13.16) by $\ell_1(U_4^*)$. We get

$$-(\lambda_2(U_1^*)-\lambda_1(U_1^*))\,\ell_1(U_2^*)\,r_2(U_1^*)b-\dot{s}_2\ell_1(U_2^*)(U_2^*-U_1^*)=0, \quad (13.20)$$

$$(\lambda_2(U_2^*)-\lambda_1(U_2^*))d-(\lambda_2(U_1^*)-\lambda_1(U_1^*))\,\ell_2(U_2^*)\,r_2(U_1^*)b$$
$$-\dot{s}_2\ell_2(U_2^*)(U_2^*-U_1^*)=0, \quad (13.21)$$

$$f=d, \quad (13.22)$$

$$-\ell_1(U_4^*)(\lambda_2(U_3^*)-\lambda_1(U_3^*))\,r_2(U_3^*)f-\dot{s}_4\ell_1(U_4^*)(U_4^*-U_3^*)=0. \quad (13.23)$$

We have used Lemma 2.2 in Eq. (13.22). Equations (13.20)–(13.23) can be solved for $(\dot{s}_2, d, f, \dot{s}_4)$ in terms of $b$. Using this solution, we obtain $\dot{s}_4-\dot{s}_2=mb$, where

$$m=\frac{\lambda_2(U_1^*)-\lambda_1(U_1^*)}{\ell_1(U_2^*)(U_2^*-U_1^*)\,\ell_1(U_4^*)(U_4^*-U_3^*)}$$
$$\times\{-\ell_1(U_4^*)\,r_2(U_3^*)\cdot\ell_2(U_2^*)\,r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*)$$
$$+\ell_1(U_4^*)\,r_2(U_3^*)\cdot\ell_1(U_3^*)\,r_2(U_1^*)\cdot\ell_2(U_2^*)(U_2^*-U_1^*)$$
$$+\ell_1(U_2^*)\,r_2(U_1^*)\cdot\ell_1(U_4^*)(U_4^*-U_3^*)\}. \quad (13.24)$$

We shall shortly verify that $m\neq 0$. (The denominator of $m$ is nonzero by the wave nondegeneracy conditions, and the bracketed expression can be rewritten to equal the expression in assumption (3) of the theorem.) Then $D\tilde{H}(U_0^*)=m\ell_2(U_1^*)\,D_1\psi(U_0^*, s_1^*)$, a nonzero vector, so that (E2) holds. (Without the simplifying assumption $k=1$, assumption (4) of the theorem would be needed.) Therefore $\mathscr{C}=\{U_0\colon \tilde{H}(U_0)=0\}$ is a smooth curve near $U_0^*$, and for $(U_0, U_n)$ near $(U_0^*, U_n^*)$, a solution of type $(R_1, RS\cdot RS, R_1, RS\cdot RS, R_1, T_6, \dots, T_n)$ exists provided $U_0$ is on the side of $\mathscr{C}$ to which this vector points.

*Step* 2. Next we consider the point $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*, s_5^*, U_5^*, s_6^* \dots, s_n^*, U_n^*)$ in $\mathbb{R}^{3n-8}$. We shall investigate the existence of nearby points $(U_0, s_1, U_1, s, U, s_5, U_5, s_6, \dots, s_n, U_n)$ that represent Riemann solutions of type $(R_1, RS\cdot RS, R_1, T_6, \dots, T_n)$. To obtain a condition for the existence of such points, we first solve the system (13.1),

$$F(U)-F(U_1)-s(U-U_1)=0, \quad (13.25)$$

$$\lambda_1(U_1)-s=0, \quad (13.26)$$

$$\lambda_1(U)-s=0, \quad (13.27)$$

for $(s_1, U_1, s, U)$ in terms of $U_0$ near $(U_0^*, s_1^*, U_1^*, s_4^*, U_4^*)$, and denote the solution $(\hat{s}_1(U_0), \hat{U}_1(U_0), \hat{s}(U_0), \hat{U}(U_0))$. This can be done because of assumption (2) of the theorem. Then, using $U_1 = \hat{U}_1(U_0)$, we solve Eqs. (13.2) and (13.4) for $(s_2, U_2)$ in terms of $U_1$. We obtain $s_2$ and $U_2$ as functions of $U_0$, which we denote $\hat{s}_2(U_0)$ and $\hat{U}_2(U_0)$. Then for $U_0$ fixed, the 1-parameter family

$$\dot{U} = F(U) - F(\hat{U}_1(U_0)) - s(U - \hat{U}_1(U_0)) \tag{13.28}$$

has transcritical and saddle-node bifurcations at $s = \hat{s}_1(U_0)$, and a second saddle-node bifurcation at $s = \hat{s}_2(U_0)$. If the parabolic curves through the saddle-node bifurcation points open to the right, the possible bifurcation diagrams of (13.28) are shown in Fig. 13.1.

Only for $U_0$ yielding the bifurcation diagram of Fig. 13.1c do we have an $RS \cdot RS$ shock from $\hat{U}_1(U_0)$ to $\hat{U}(U_0)$. For $U_0$ yielding the bifurcation diagram of Fig. 13.1a we have a generalized $RS \cdot RS$ shock from $\hat{U}_1(U_0)$ to $\hat{U}(U_0)$.

Once $U = \hat{U}(U_0)$ is found, the remainder of the Riemann solution is obtained by solving for $(s_5, U_5, ..., U_{n-1}, s_n)$ in terms of $(U, U_n)$. If the lower parabola in the diagrams of Fig. 13.1 opens to the right as shown, i.e., if $\ell_1(U_2^*)(U_2^* - U_1^*) > 0$, the solution actually represents a wave sequence of the desired type if and only if $\hat{s}_1(U_0) < \hat{s}_2(U_0)$. (If $\ell_1(U_2^*)(U_2^* - U_1^*) < 0$, we need $\hat{s}_1(U_0) > \hat{s}_2(U_0)$.) We therefore study the function $\tilde{H}(U_0, U_n) := \hat{s}_2(U_0) - \hat{s}_1(U_0) = \hat{s}_2(U_0) - \hat{s}(U_0)$.

We calculate $D_{U_0}\tilde{H}(U_0^*, U_n^*)\,\dot{U}_0$ by linearizing Eqs. (13.1), (13.2), (13.4), (13.25), (13.26), (13.27) at $(U_0^*, s_1^*, U_1^*, s_2^*, U_2^*, s^*, U^*)$ and calculating $\dot{s}_2 - \dot{s}$ in terms of $\dot{U}_0$.

The linearized system is (13.12), (13.13), (13.14),

$$\begin{aligned} (DF(U_4^*) - s_4^* I)\,\dot{U} - (DF(U_1^*) - s_4^* I)\,\dot{U}_1 - \dot{s}(U_4^* - U_1^*) &= 0, \\ D\lambda_1(U_1^*)\,\dot{U}_1 - \dot{s} &= 0, \\ D\lambda_1(U_4^*)\,\dot{U} - \dot{s} &= 0. \end{aligned} \tag{13.29}$$

We make the substitution (13.17) and multiply Eq. (13.13) by $\ell_1(U_2^*)$ to obtain Eq. (13.20). Then we multiply Eq. (13.29) by $\ell_1(U_4^*)$ to obtain

$$-\ell_1(U_4^*)(\lambda_2(U_1^*) - \lambda_1(U_1^*))\, r_2(U_1^*) b - \dot{s}\ell_1(U_4^*)(U_4^* - U_1^*) = 0.$$

From these two equations, we obtain that $\dot{s}_2 - \dot{s} = nb$, where

$$\begin{aligned} n = {} & -\frac{\lambda_2(U_1^*) - \lambda_1(U_1^*)}{\ell_1(U_4^*)(U_4^* - U_1^*)\,\ell_1(U_2^*)(U_2^* - U_1^*)} \\ & \times \{ -\ell_1(U_4^*)\, r_2(U_1^*) \cdot \ell_1(U_2^*)(U_2^* - U_1^*) \\ & + \ell_1(U_2^*)\, r_2(U_1^*) \cdot \ell_1(U_4^*)(U_4^* - U_1^*) \}. \end{aligned} \tag{13.30}$$

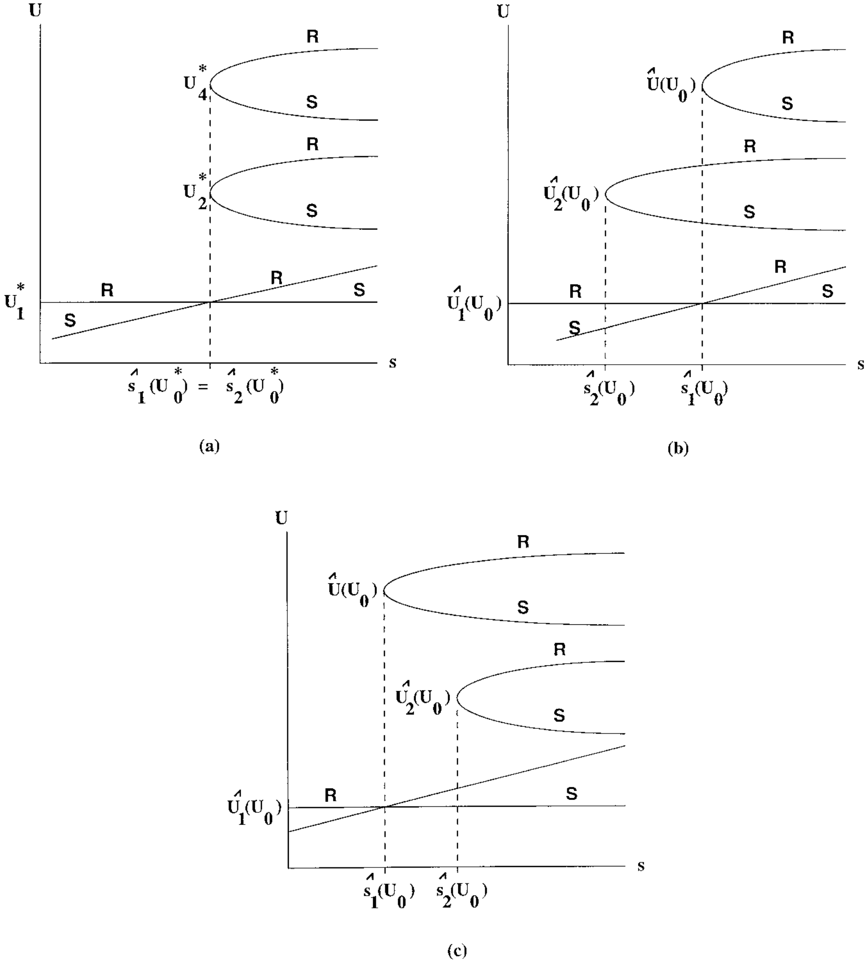


**FIG. 13.1.** Bifurcation diagrams for $\dot{U} = F(U) - F(\hat{U}_1(U_0)) - s(U - \hat{U}_1(U_0))$ for $U_0$ near $U_0^*$. Only in case (c) is there a repeller-saddle to repeller-saddle connection from $U_1(U_0)$ to $\hat{U}(U_0)$.

Assumption (3) of the theorem implies that $n \neq 0$. Then $D\tilde{H}(U_0^*) = n\,\ell_2(U_1^*)D_1\psi(U_0^*, s_1^*)$, a nonzero vector. (Without the simplifying assumption $k = 1$, assumption (4) of the theorem would be needed.)

We claim that

$$m = -\frac{\ell_1(U_4^*)(U_4^* - U_1^*)}{\ell_1(U_4^*)(U_4^* - U_3^*)}\,n. \tag{13.31}$$

Then by assumption (2), $m \neq 0$, which completes Step 1.

To prove (13.31), we must show that the bracketed expressions in (13.24) and (13.30) are equal. After some rearrangement, we must show that

$$\begin{aligned}
&-\ell_1(U_4^*)\, r_2(U_3^*)\cdot\ell_2(U_2^*)\, r_2(U_1^*)\ \cdot\ell_1(U_2^*)(U_2^*-U_1^*)\\
&\qquad+\ell_1(U_4^*)\, r_2(U_3^*)\cdot\ell_1(U_2^*)\, r_2(U_1^*)\ \cdot\ell_2(U_2^*)(U_2^*-U_1^*)\\
&\qquad+\ \ell_1(U_4^*)\, r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*)\\
&\quad=-\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_4^*)(U_4^*-U_3^*)\\
&\qquad+\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_4^*)(U_4^*-U_1^*).
\end{aligned}\tag{13.32}$$

The right hand side is

$$\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_4^*)(U_3^*-U_1^*).\tag{13.33}$$

On the left hand side of Eq. (13.32), we subtract and add

$$\ell_1(U_4^*)\, r_1(U_3^*)\cdot\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*),\tag{13.8}$$

then combine terms to obtain

$$\begin{aligned}
&-\ell_1(U_4^*)\{\ell_2(U_2^*)\, r_2(U_1^*)\cdot r_2(U_3^*)\\
&\qquad+\ell_1(U_2^*)\, r_2(U_1^*)\cdot r_1(U_3^*)\}\cdot\ell_1(U_2^*)(U_2^*-U_1^*)\\
&\qquad+\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_4^*)\{\ell_1(U_2^*)(U_2^*-U_1^*)\cdot r_1(U_3^*)\\
&\qquad+\ell_2(U_2^*)(U_2^*-U_1^*)\cdot r_2(U_3^*)\}\\
&\qquad+\ell_1(U_4^*)\, r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*)\\
&\quad=-\ell_1(U_4^*)\, r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*)\\
&\qquad+\ell_1(U_2^*)\, r_2(U_1^*)\cdot\ell_1(U_4^*)(U_2^*-U_1^*)\\
&\qquad+\ell_1(U_4^*)\, r_2(U_1^*)\cdot\ell_1(U_2^*)(U_2^*-U_1^*),
\end{aligned}$$

which equals (13.33).

*Step* 3. It is easy to see that $\{U_0 : \tilde{H}(U_0)=0\}$, where $\tilde{H}$ comes from Step 2, is precisely the curve $\mathscr{C}$ defined in Step 1. For $(U_0, U_n)$ near $(U_0^*, U_n^*)$ we have the following conclusions:

(1) $\ell_1(U_2^*)(U_2^*-U_1^*)>0$. Solutions of type $(R_1, RS\cdot RS, R_1, T_6, ..., T_n)$ exist on the side of $\mathscr{C}$ to which $n$ points. By Eq. (13.31) the join is regular (resp. folded) if $\ell_1(U_4^*)(U_4^*-U_1^*)\,\ell_1(U_4^*)(U_4^*-U_3^*)$ is positive (resp. negative).

(2) $\ell_1(U_2^*)(U_2^* - U_1^*) < 0$. Solutions of type $(R_1, RS \cdot RS, R_1, T_6, ..., T_n)$ exist on the side of $\mathscr{C}$ opposite that to which $n$ points. Thus the join is regular (resp. folded) if $\ell_1(U_4^*)(U_4^* - U_1^*)\, \ell_1(U_4^*)(U_4^* - U_3^*)$ is negative (resp. positive). ∎

*Remark.* Assumption (3) is analagous to assumption (2) of Theorem 10.1. It is required so that the generalized shock from $U_k^*$ to $U_{k+3}^*$ ("generalized" because the two points are connected by a chain of two connections, rather than a single connection) will nevertheless satisfy nondegeneracy condition (B2) for $RS \cdot RS$ shocks.

*Remark.* Assumption (3) has the following geometric interpretation. The triplets $(U_1, s_2, U_2)$ such that there is a shock of type $RS \cdot RS$ from $U_1$ to $U_2$ with speed $s_2$ form a curve $\mathscr{D}$ through $(U_1^*, s_2^*, U_2^*)$: the solutions of Eqs. (13.2)–(13.4) . Similarly, the triplets $(U_3, s_4, U_4)$ such that there is a shock of type $RS \cdot RS$ from $U_3$ to $U_4$ with speed $s_4$ form a curve $\tilde{\mathscr{D}}$ through $(U_3^*, s_4^*, U_4^*)$: the solutions of Eqs. (13.8)–(13.12). The curve $\mathscr{D}$ projects to curves $\mathscr{D}_1$ and $\mathscr{D}_2$ through $U_1^*$ and $U_2^*$ respectively; the curve $\tilde{\mathscr{D}}$ projects to curves $\mathscr{D}_3$ and $\mathscr{D}_4$ through $U_3^*$ and $U_4^*$ respectively. Assumption (3), which, as the proof shows, is equivalent to requiring that the bracketed expression in Eq. (13.24) be nonzero, says that $\mathscr{D}_2$ and $\mathscr{D}_3$ meet transversally at $U_2^* = U_3^*$. As in the previous sections, this is a natural geometric requirement for codimension-one Riemann solutions of the desired type. It is used in both steps 1 and 2 of the proof.

## 14. FINAL REMARKS

In this section we make some remarks that relate the dependence of the boundary type of a codimension-one Riemann solution on its position in the wave sequence, as discussed at the end of Section 3; Table III, which gives the possible boundary types of the classical missing rarefaction solutions; and the wave curve mapping regularity assumptions that appear in the statements of Theorems 6.1, 8.1–9.1, and 11.1–13.1.

The reader will note that in all six classical missing 1-rarefaction cases that can be intermediate boundaries—i.e., all classical missing 1-rarefaction cases except those in which the missing 1-rarefaction is immediately followed by an $RS \cdot RS$ shock wave—a regularity hypothesis on a backward wave curve mapping $\tilde{U}_m(U_n, \tau)$ is needed. When the missing 1-rarefaction is followed at some point in the wave sequence by a 1-rarefaction, this hypothesis cannot be satisfied: the backward wave curve mapping whose regularity is needed is actually independent of $U_n$ (up to reparameterization in $\tau$). In this situation, we expect a $U_L$-boundary rather than an intermediate boundary.

In two of these six cases, the missing 1-rarefaction is the first wave in the Riemann solution, and in two it is the second. However, in two of the cases—those in which the missing 1-rarefaction is immediately preceded by an $RS \cdot RS$ shock wave—the number of waves that precede the missing 1-rarefaction is arbitrarily large. In these two cases regularity of the forward wave curve mapping $U_{k+1}(U_0, s_{k+1})$ at $(U_0^*, s_{k+1}^*)$ must be assumed. However, if a 2-rarefaction precedes the missing 1-rarefaction somewhere in the wave sequence, this hypothesis cannot be satisfied: the forward wave curve mapping whose regularity is needed is independent of $U_0$. In this situation, we expect the dual of a $U_L$-boundary rather than an intermediate boundary. If the missing 1-rarefaction is both preceded by a 2-rarefaction and followed by a 1-rarefaction, then neither regularity hypothesis can be satisfied, and we expect to have an $F$-boundary.

The final missing 1-rarefaction case treated in this paper—the missing 1-rarefaction is both preceded and followed by $RS \cdot RS$ shock waves—is a little different. Since the second $RS \cdot RS$ shock wave must be followed by a 1-rarefaction, we cannot have an intermediate boundary. The reader will note that to show that we have a $U_L$-boundary, regularity of the forward wave curve mapping $U_{k+1}(U_0, s_{k+1})$ at $(U_0^*, s_{k+1}^*)$ must be assumed. If a 2-rarefaction precedes the missing 1-rarefaction somewhere in the wave sequence, this hypothesis cannot be satisfied, and we expect to have an $F$-boundary.

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